An introduction to algebraic topology and covering spaces

Spring school in Pristina, Kosovo

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Introduction

This document is the content of a course given in Pristina, Kosovo in order to familiarize students with some topics of research. I choose this topic, first because I am really interested with such fields, and secondly because the intuition plays an important role, and thus even if the reader is not completely familiar with some notion of general topology, I think he will be able to understand at least the basic idea and the sketch of the proofs behind the theory. I would underline one more time the importance, in my point of view at least, to have a representation of what we do, that is, even if no pictures are included in the present document, it will be of a great interest for the reader to follow the proofs or the constructions with the help of some drawing.

As say previously, we will use in this document the basic notion of general topology and even if it will not be clearly , we will always mean by "space" a "topological space" and by a map between spaces a continuous one. Moreover, when a group acts on a topological space, this action will also assumed to be continuous. We will also use, sometimes, the basic language of category theory, but if the reader do not know what a category is, then he just has to skip these few notes and continue without problem.

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1 Exponential and logarithm

Let us consider the exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$.

We know that this map is surjective, is a groups homomorphism and is $2i\pi$-periodic.

But what about the existence of a complex logarithm $\mathbb{C}^* \to \mathbb{C}$ which would be the inverse of the exponential map ?

Of course such a map could not exist, because the exponential map is not injective on $\mathbb{C}$. But does such a complex logarithm exists at least locally ?

Definition 1. Let A be a subset of $\mathbb{C}^*$.

We say that a continuous map $f : A \to \mathbb{C}$ is a *choice of a logarithm on* A if the composition $\exp \circ f : A \to \mathbb{C}^*$ is equal to the identity of A.

In other words, the question of finding locally a complex logarithm is the same as to find some subset A where we have a choice of a logarithm.

If z is a non-zero complex number, let chose $\arg(z)$ an argument of z. Then we have the equivalences

$$\begin{aligned} z = \exp(t) &\Leftrightarrow |z| \exp(i \arg(z)) = \exp(t) \\ &\Leftrightarrow \exists k \in \mathbb{Z}, |z| = \exp(Re(t)) \text{ and } \arg(z) = Im(t) + 2\pi k \\ &\Leftrightarrow \exists k \in \mathbb{Z}, t = \ln|z| + i \arg(z) + 2i\pi k \end{aligned}$$

As we see, the problem of inverting a logarithm come from the fact that an argument exists only modulo 2π. In order to avoid this problem, we can chose $A = \mathbb{C}^* \backslash \{z \in \mathbb{C}, Re(z) \leq 0\}$ and for argument the continuous map $\arg : A \to]-\pi, \pi[$. Then for each k, the map

$$z \mapsto \ln|z| + i \arg(z) + 2i\pi k$$

defines a choice of a logarithm on A. Moreover, all of this image in $\mathbb{C}$ are pairwise disjoint and recovered $\exp^{-1}(A)$. In a more geometric point of view, when we restrict the exponential map from $\exp^{-1}(A)$ to A, then $\exp^{-1}(A)$ is nothing else than $\mathbb{Z}$ copies of A.

But of course we could have make an other choice for A, for example

$$\mathbb{C}^* \backslash \{z \in \mathbb{C}, Re(z) \geq 0\}$$

or

$$\mathbb{C}^* \backslash \left\{ z \in \mathbb{C}^*, arg(z) \equiv \frac{\pi}{4} [2\pi] \geq 0 \right\}$$

and the previous construction will still be true.

Remark. Let $f : [0, +\infty[\to \mathbb{C}$ an injective continuous map such that $f(0) = 0$ and the norm $|f(t)|$ tends to $+\infty$ when t tends to $+\infty$. Then a choice of a logarithm exists on $\mathbb{C} \backslash f([0, +\infty[)$.

What we have done here is the idea behind a covering space and thus it will be good for the reader to keep this example in mind when reading.

2 Covering spaces

Definition 2. Let B be a topological space.

- A *covering space of* B is a couple (X, p) where X is a topological space and p is a map from X to B which satisfies the following property :

  for all point b in B there exists an open neighborhood U of b in B, a non-empty discrete space F and an homeomorphism $\phi : p^{-1}(U) \to U \times F$ such that the following diagram commutes

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "p_{|p^{-1}(U)}"'] & & U \times F \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

where $pr_1 : U \times F \to U$ is the first projection map.

In such a situation, we say that B is the *base* of the covering, X is the *total space* and $p^{-1}(b)$ (which is isomorphic to F) the *fibre over* b.

Note that because we assume in our definition that F is non-empty, then such a map p is always surjective.

- A *finite covering space of* B is a covering space (X, p) of B such that for every point b in B, the fibre over b of the covering is a finite set. Moreover, if all of these fibrers have the same cardinality, say d, then we say that (X, p) is a *finite covering space of* B *of degree* d or a 2-*sheets covering space of* B.

Remark. We will often write just p to denote the covering space (X, p) of B.

Examples. 1. Let $f : X \to Y$ an homeomorphism. Then (Y, f) is a covering space of Y.

2. Let B and F be two topological spaces with F discrete. Then $(B \times F, pr_1)$, where pr_1 denotes the first projection map, is a covering space of B. We say that this covering is the *trivial covering space of* B *of fibre* F.

   A covering space which is isomorphic to such a covering will be called *trivial* (see the next definition for the notion of isomorphism).

3. Let $\mathbb{S}^1$ denote the unit circle of $\mathbb{R}^2$. Then $(\mathbb{R}, \exp)$ is a covering space of $\mathbb{S}^1$.

4. We also have that $(\mathbb{C}, \exp)$ is a covering space of $\mathbb{C}^*$.

5. Let $\mathbb{Z}$ acts on $\mathbb{R}$ by addition. We then form the quotient space $\mathbb{Z}\backslash\mathbb{R}$ and we have a natural quotient map $p : \mathbb{R} \to \mathbb{Z}\backslash\mathbb{R}$.

   Then $(\mathbb{R}, p)$ is a covering space of $\mathbb{Z}\backslash\mathbb{R}$.

   Remark that this example is quite the same as the second one because we have an homeomorphism $\mathbb{S}^1 \simeq \mathbb{Z}\backslash\mathbb{R}$.

6. Let p be the map from $]0, 3\pi[$ to $\mathbb{S}^1$ defines by the association $t \mapsto e^{it}$. Then p is a local homeomorphism but $(]0, 3\pi[, p)$ is not a covering space of $\mathbb{S}^1$.

Exercise 1. Let $p : X \to B$ and $p' : X' \to B'$ be two covering spaces.
Show that the map from $X \times X'$ to $B \times B'$ which sends (x, x') to $(p(x), p'(x'))$ is a covering space of $B \times B'$.

Exercise 2. Let $p : X \to B$ be a covering space with B connected.
Show that all the fibers are homeomorphic to a same non-empty discrete topological space.

Definition 3. Let B be a topological space and (X, p) and (Y, q) two covering spaces of B.

- A *morphism of covering spaces of* B *from* (X, p) *to* (Y, q) is a continuous map $f : X \to Y$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

- We say that a morphism $f : (X, p) \to (Y, q)$ of covering spaces of B is an *isomorphism* if there exists a morphism of covering spaces $g : (Y, q) \to (X, p)$ such that the topological maps $g \circ f$ and $f \circ g$ are the identity maps of X and Y respectively.

We have built with these definitions a category, written $Cov(B)$, of covering spaces of B. We also have a full subcategory, $FCov(B)$, which objects are the finite covering spaces of B.

Exercise 3. Let $p : Y \to X$ and $q : X \to B$ be two covering spaces and assume that the covering q is finite.

1. Show that the composition map $q \circ p : Y \to B$ is a covering space of B. We will denote this covering as $q_!(p)$.
2. If moreover p and q are finite covering of degree d and d' respectively, what is the degree of $q_!(p)$?
3. (this question uses the language of category theory)

   Show that $q_! : Cov(X) \to Cov(B)$ is a functor. Moreover, if $q' : X' \to X$ is another finite covering, show that we get a natural transformation

$$(q \circ q')_! \simeq q_! \circ q'_!.$$

Exercise 4. Let $p : X \to B$ be a covering space and $f : B' \to B$ a continuous map between topological spaces.

Define

$$X \times_B B' := \{(x, b') \in X \times B', p(x) = f(b')\}$$

and denote by q the map from $X \times_B B'$ to B' which sends a couple (x, b') to b'.

1. Show that q is a covering space of B'. It is call the *base change of p by f* or the *pull-back of p by f* and is often written as $f^*(p)$.
2. If moreover p is a finite covering of degree d, what is the degree of $f^*(p)$?
3. (this question uses the language of category theory)

   Show that $f^* : Cov(B) \to Cov(B')$ is a functor. Moreover, if $f' : B'' \to B'$ is another continuous map between topological spaces, show that we get a natural transformation

$$(f \circ f')^* \simeq f'^* \circ f^*.$$

Exercise 5. (this exercise uses the language of category theory)

We use the notations and results of exercices 3. and 4..

Show that if $f : B' \to B$ is a finite covering of B, then the functor $f_!$ is left adjoin to the functor f^*.

Now we want to generalise our previous example 5. For this, let G be a group which acts on the left on a topological space X. We say that the group acts *evenly* if for each $x \in X$, there exists an open neighborhood U of x such that the gU, for $g \in G$, are pairwise disjoint.

We then get a covering space by the following proposition :

Proposition 4. *Let G be a group which acts evenly on the left on a topological space X.*

Then (X, p) is a covering space of $G\backslash X$, where $p : X \to G\backslash X$ denotes the canonical quotient map.

Remark. In the literature, it is possible to find the term „properly discontinuous" to speak either about an evenly action as defined before, or for a generalization of it. Nevertheless, the goal of these hypothesis is the covering result previously expressed and the statement here will be enough for our purposes.

Proof. Let x be a point in X and U an open neighborhood of x in X which satisfies the previous „evenly action condition". Then $p(U)$ is an open neighborhood of $p(x)$ and by construction of U, we have an homeomorphism $p^{-1}(p(U)) \simeq p(U) \times G$. □

Examples. 1. Let G be the subgroup of $(\mathbb{R}, +)$ generates by 2π. We then get the exponential covering space $\mathbb{R} \to \mathbb{S}^1 \simeq G\backslash\mathbb{R}$.

More generally, we can consider the action of $\mathbb{Z}^n$ to $\mathbb{R}^n$ and thus obtain a covering

$$\mathbb{R}^n \to \mathbb{Z}^n\backslash\mathbb{R}^n \simeq (\mathbb{S}^1)^n =: \mathbb{T}^n.$$

2. If $n \geq 1$ is a positive integer, the canonical map $\mathbb{S}^n \to \mathbb{RP}^n$ is the covering space of degree 2 given by the antipodal action $x \mapsto -x$ on the sphere,

3. Let $\mathbb{U}_n := \{z \in \mathbb{C}, z^n = 1\}$, then this group acts on $\mathbb{S}^1$ and the associated covering space is the map $z \mapsto z^n$ from $\mathbb{S}^1 \to \mathbb{S}^1 \simeq \mathbb{U}_n \backslash \mathbb{S}^1$.

We now deal to the important notion of lifting.

Definition 5. Let B be a topological space, $p : X \to B$ a covering and $f : B' \to B$ a continuous map.
A *lifting of f for p* is a continuous map $g : B' \to X$ such that the following diagram commutes.

$$\begin{tikzcd} & X \arrow[d, "p"] \\ B' \arrow[ur, "g"] \arrow[r, "f"'] & B \end{tikzcd}$$

A *section of the covering* p is a lifting of the identity of B for p.

Remark. Note that if $p : B \times F \to F$ is a trivial covering, then a section of p always exists. Indeed, we just have to fix an element f in F and to define a map $\overline{p}$ by the association $b \mapsto (b, f)$.

As by definition a covering is locally isomorphic to a trivial one, that means that a section always exists locally.

Exercise 6. Let $p : X \to B$ be a covering space and $f : B' \to B$ a continuous map.
Show that we have a bijection between the liftings of f for p and the section of the pull-back covering $f^*(p)$ defined in exercise 4.

First an important lemma :

Lemma 6. *Let $p : X \to B$ be a covering and f and g two maps from a connected topological space B' to X such that $p \circ f = p \circ g$.*
Assume that there exists a point in B' where f and g coincide. Then $f = g$.

Proof. Consider the set A of the points of B' where f and g coincide. By assumption, this set is not empty and let x be an element of A. Let $y = p \circ f(x)$ then by definition of a covering, there exists a trivialisable open neighborhood of y in B for the covering p, that means in particular that $p^{-1}(U)$ is homeomorphic to $U \times F$ where F is a non-empty discrete topological space. Now consider the element $a \in F$ such that the image of x by f (or equivalently by g) is in $U \times \{a\} \subset U \times F$. Then by continuity, there exists an open neighborhood V of x in B' such that the image of V by f and g are included in the previous plate $U \times \{a\}$. But then the equality $p \circ f = g \circ f$ shows that f and g must coincide on V and thus the subset A of B' is open.
The same reasoning shows that the set of the points of B' where f and g do not coincide is also open. That means that A is also closed and thus is equal to B' by connectedness. $\square$

Let $p : X \to B$ be a covering and denote by $Aut(p)$ the group of automorphisms of p, that means isomorphisms from p to p.
We then have the following corollary :

Corollary 7. *Let $p : X \to B$ be a covering space with X connected.*
Then the group $Aut(p)$ acts evenly on X.

Proof. Let x be a point in X. We note $b = p(x)$ and we choose an open neighborhood U of b in B which is trivialisable for p. We then have that $p^{-1}(U)$ is homeomorphic to $U \times F$ for a non-empty discrete topological space F. Let f be the plate containing x, that means that $x \in U \times \{f\} \subset U \times F \simeq p^{-1}(U)$ and we note $V = U \times \{f\}$.
This set V is an open neighborhood of x in X and we just have to prove now that if $V \cap gV \neq \varnothing$ for a certain element $g \in Aut(p)$, then $g = id_X$. But if $y \in V \cap gV$ then y must be fixed by g : indeed its image gy is as y a point over $p(y)$ and the fact that $gy \in V$ shows they belong to the same plate, thus are equal. But now, the previous lemma applied to g and id_X gives us that these maps are the same. $\square$

In particular, that means that we get a covering space

$$X \to Aut(p)\backslash X$$

and we see that we get a factorisation of p by this map, in other words there exists a map $Aut(p)\backslash X \to B$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "p"'] & Aut(p)\backslash X \arrow[dl] \\ B & \end{tikzcd}$$

Definition 8. We say that a covering $p : X \to B$ is a *Galois covering* if X is a connected space and if the previous map $Aut(p)\backslash X \to B$ is an homeomorphism.

Remark. Note the analogy with a Galois extension for fields. Indeed, if $m : K \to L$ is a fields extension and if we note $G = Aut(L|K)$ the group of the K-automorphisms of L, then it is possible to form a field L^G defines by

$$L^G := \{x \in L, \forall g \in G, gx = x\}$$

and we get a natural fields extension $L^G \to L$ together with a factorisation of m of the form

$$K \to L^G \to L.$$

And we say in this context that the extension $K \to L$ is Galois if the previous map $K \to L^G$ is an isomorphism.

We will see that the analogy go further.

Exercise 7. Let $p : X \to B$ be a covering space with Y connected.

Show that p is a Galois covering if and only if the group $Aut(p)$ acts transitively on each fibres of p.

Proposition 9. *Let G be a group which acts evenly on a connected topological space X and we denote by $p : X \to G\backslash X$ the associated covering space.*

Then we have a groups isomorphism $G \simeq Aut(p)$ and the covering p is Galois.

Proof. First note that if g is an element of G, then the map $X \to X$ given by $x \mapsto gx$ defines an automorphism of p and thus we have a natural inclusion map $G \to Aut(p)$.

Let φ be an automorphism of p and x an element of X. We know that $\varphi(x)$ is an element over $p(x)$ and hence there exists an element g in G such that $\varphi(x) = gx$. Then g and φ are two automorphisms of p which coincide on x, that means there are equal because X is connected. The fact that p is Galois is now clear. $\square$

We will now give without proof the main statement for Galois coverings (for this, see [9] for example). To underline the analogy with the main statement in field theory, we will recall it first :

Theorem 10 (Main theorem of Galois theory for finite extension)**.** *Let $K \to L$ be a finite Galois extension with Galois group G.*

The maps $M \mapsto H := Aut(L|M)$ and $H \mapsto M := L^H$ are inverse from each other and then realise a bijection between the subfield of L containing K and the subgroup of G.

Moreover, the extension $M \to L$ is always Galois, and the extension $K \to M$ is Galois if and only if the associated subgroup H of G is normal, in which case we have an isomorphism

$$Gal(M|K) \simeq G/H.$$

Theorem 11. *Let $p : X \to B$ be a Galois covering and denote by G the group $Aut(p)$.*

For each subgroup H of G, we have the existence of a commutative triangle

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "p"'] & H\backslash X \arrow[d, "q"] \\ & B \end{tikzcd}$$

where q is a covering (and π the canonical Galois covering).

Reciprocally, for each commutative diagram of the form

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "p"'] & Y \arrow[d, "q"] \\ & B \end{tikzcd}$$

where q is a covering, then π is a Galois covering and we can associate the group $Aut(\pi)$.

Then by these association, we define two applications which are inverse from each other. In other terms, we have a bijection between the subgroup of H and the covering spaces of B which may be covered by p.

Finally, the covering q is Galois if and only if the associated subgroup H of G is normal. We then have that the group $Aut(q)$ is isomorphic to the quotient group G/H.

3 First homotopy group of a topological space

We will denote by Δ^0 the topological space which consists of just a single point and by Δ^1 the segment $[0, 1]$ of $\mathbb{R}$. We obviously have two privilegiate continuous maps s and t

$$s, t : \Delta^0 \to \Delta^1,$$

sending the single point of Δ^0 to 0 and 1 respectively.

Let X be a topological space. We first note that it is possible to identify points of X which maps from Δ^0 to X.

Definition 12. A *path in* X is a map from $\sigma : \Delta^1 \to X$.

With the previous identification, we say that the points $\sigma \circ s$ and $\sigma \circ t$ are respectively the *source* and the *target of the path* σ.

A *loop in* X is a path in X where source and target coincide.

We will denote by $\Pi_1(X, s, t)$ the set of paths in X with source s and target t. In the case where $t = s$, we will simply write $\Pi_1(X, s)$ for this set.

If σ is in $\Pi_1(X, s, t)$, then it is possible to form an other path, written σ^{-1}, given by the association $u \mapsto \sigma(1 - u)$. We see that this operation is an involution and inverses source and target, that means σ^{-1} is an element of $\Pi_1(X, t, s)$. This path is called the *inverse path of* σ.

If now τ is an element of $\Pi_1(X, t, t')$, then we can also form a path, written $\tau \circ \sigma$ or simply $\tau\sigma$, given by $\tau\sigma(u) = \sigma(2u)$ if $0 \leq u \leq \frac{1}{2}$ and $\tau\sigma(u) = \sigma(2u - 1)$ if $\frac{1}{2} \leq u \leq 1$. The path $\tau\sigma$ is an element of $\Pi_1(X, s, t')$ called the *composition of* σ *by* τ.

Remark. In the literature, it is possible to find the notation $\sigma\tau$ for the composition define before. I prefer to take the convention $\tau\sigma$ in this document because it coincides with the convention of composition of morphisms.

Definition 13. Let σ and σ' be two elements in $\Pi_1(X, s, t)$.

A *homotopy between* σ *and* σ' is a map $H : \Delta^1 \times \Delta^1 \to X$ such that

$$\begin{cases} H(., 0) = \sigma, \\ H(., 1) = \sigma', \\ H(0, .) = s, \\ H(1, .) = t. \end{cases}$$

The next two results are left as exercises :

Lemma 14. *The relation „being homotopic" is an equivalence relation on* $\Pi_1(X, s, t)$.

We will then denote by $\pi_1(X, s, t)$ the quotient of $\Pi_1(X, s, t)$ by this equivalence relation (or simply by $\pi_1(X, s)$ in the case where $t = s$).

Lemma 15. *The inverse and composition maps*

$$\begin{array}{c} \Pi_1(X,s,t) \to \Pi_1(X,t,s) \\ \sigma \mapsto \sigma^{-1} \end{array} \quad \text{and} \quad \begin{array}{c} \Pi_1(X,s,t) \times \Pi_1(X,t,t') \to \Pi_1(X,s,t') \\ (\sigma,\tau) \mapsto \tau\sigma \end{array}$$

induces maps

$$\pi_1(X,s,t) \to \pi(X,t,s) \quad \text{and} \quad \pi_1(X,s,t) \times \pi(X,t,t') \to \pi_1(X,s,t')$$

respectively.

Proposition 16. *In the case where* $s = t = t'$*, then the two previous maps defines a group structure on the set* $\pi_1(X, s, s)$.

Proof. Only the associativity condition do not result from the previous lemma. We left this verification to the reader. □

Definition 17. The group $\pi_1(X, s)$ is called the *fundamental group of* X *in* s, or *Poincaré group of* X *in* s or *first homotopy group of* X *in* s. It was introduced for the first time by the mathematician Henri Poincaré in 1895.

Remark. Of course, like the name „first homotopy group" suggests, there exists a theory of higher homotopy groups. But we will not deals with this problem in this document.

The fundamental group is an important notion in topology because, as we will see, it gives us an homotopical invariant. We will be more precise latter.

If σ is a path in X from a source s to a target t, then we see that it is possible to construct a groups isomorphism $\pi_1(X, s) \simeq \pi_1(X, t)$. Indeed, to a loop in X with endpoint s, we can consider the loop $\sigma s \sigma^{-1}$ in Y with endpoint t and verifies that this association induces a desire isomorphism. In particular, if X is path connected, then the fundamental group of X do not depends on any basepoint and we will write it simply $\pi_1(X)$.

We give now the first non-trivial example, that is the case of the circle. We will prove the statement latter but we enounce it now in order to use freely this result to catch more examples.

Theorem 18. *The fundamental group of the sphere* $\mathbb{S}^1$ *is isomorphic to* $\mathbb{Z}$.

If $f : X \to Y$ are two topological spaces and x and y two points in X and Y respectively such that $f(x) = y$, then the composition by f transforms any loop in X with endpoint x to a loop in Y with endpoint y. In other words, we get an application

$$\Pi_1(X, x) \to \Pi_1(Y, y).$$

Furthermore, this application induces a groups homomorphism

$$f_* : \pi_1(X, x) \to \pi_1(Y, y).$$

In a more categorist language, we get a functor π_1 from the category of pointed topological spaces to the category of groups. Moreover, if $g : (Y, y) \to (Z, z)$ is another map between pointed topological spaces, then we get a natural transformation

$$g_* \circ f_* \simeq (g \circ f)_*.$$

Exercise 8. Let B and B' be two topological spaces and b (respectively b') a point in B (respectively B').

Show that the canonical map

$$\pi_1(B \times B', (b, b')) \to \pi_1(B, b) \times \pi_1(B', b')$$

defined by the projections maps $B \times B' \to B$ and $B \times B' \to B'$ is an isomorphism.

In particular, the fundamental group of the n-torus $\mathbb{T}^n := (\mathbb{S}^1)^n$ is $\mathbb{Z}^n$.

In order to calculate some fundamental groups of spaces, we will now generalise the notion of homotopy of paths.

Definition 19. Let $f, g : X \to Y$ be two continuous maps between topological spaces.

A *homotopy between f and g* is a map $H : X \times \Delta^1 \to Y$ such that

$$\begin{cases} H(., 0) = f, \\ H(., 1) = g. \end{cases}$$

If A is a subset of X, then we say that the homotopy H is an *homotopy relatively to A* if for each $t \in \Delta^1$, the restriction at A of the map $H(., t)$ is the identity of A.

Remark. The homotopy of paths describe before thus corresponds to the homotopy in this sense relatively to the endpoints $A = \{0, 1\}$.

Proposition 20. *The relation „being homotopic" (or „being homotopic relatively to a subspace A") is an equivalence relation on the set of continuous maps from X to Y.*

Definition 21. We say that a topological space X is *contractible* if it is non-empty and if the identity of X is homotopic to a constant map from X to X.

For example, a non-empty convex subspace of $\mathbb{R}^n$ is a contractible space.

Exercise 9. Let X be a topological space and define CX by the quotient of $X \times \Delta^1$ by the relation $(x, 1) \sim (y, 1)$ for every elements x and y in X.

Show that X is a contractible space.

Exercise 10. Show that the fundamental group of a contractible topological space is trivial.

Definition 22. Let X and Y be two topological spaces and $f : X \to Y$ a continuous map.

We say that f is an *equivalence of homotopy* if there exists a map $g : Y \to X$ such that the compositions $g \circ f$ and $f \circ g$ are homotopic to the identity of X and Y respectively.

We say that X and Y have the same *homotopy type* if there exists an equivalence of homotopy between them.

This notion is important because we will see that two paths-connected topological spaces with the same homotopy type have necessarily isomorphic fundamental groups. That means in particular that the data of the first fundamental group permits us to explicit spaces which are not homeomorphic.

Exercise 11. Verify the following results :

1. a contractible space has the same homotopy type as the space Δ^0,
2. if $n \geq 1$ is a positive integer, then the n-sphere $\mathbb{S}^n$ has the same homotopy type as $\mathbb{R}^{n+1} \backslash \{0\}$,
3. if X and Y are two topological spaces with Y contractible and if y is an arbitrary element of Y, then $X \times Y$ has the same homotopy type as $X \times \{y\}$,
4. let define the Möbius band M as the quotient of the space $[0, 1] \times [-1, 1]$ by the relation $(0, s) \sim (1, -s)$. Then M has the same homotopy type as the circle $\mathbb{S}^1$.

Proposition 23. *Let $f, g : X \to Y$ be two homotopic continuous maps and x a point in X.*

Then there exists a groups isomorphism $u : \pi_1(Y, f(x)) \to \pi_1(Y, g(x))$ such that the following triangle commutes.

$$\begin{tikzcd} & \pi_1(Y, f(x)) \arrow[dd, "u"] \\ \pi_1(X, x) \arrow[ur, "f_*"] \arrow[dr, "g_*"'] & \\ & \pi_1(Y, g(x)) \end{tikzcd}$$

Moreover, if f and g are homotopic relatively to the point $\{x\}$, then $f_* = g_*$.

Proof. Let H be an homotopy between f and g and τ the path in Y between $f(x)$ and $g(x)$ given by the association $t \mapsto H(x, t)$. Then the map

$$\Pi_1(Y, f(x)) \to \Pi_1(Y, g(x))$$
$$\sigma \mapsto \tau\sigma\tau^{-1}$$

induces a groups isomorphism $u : \pi_1(Y, f(x)) \to \pi_1(Y, g(x))$.

We will now show that u is our desire isomorphism, that is that $g_* = u \circ f_*$. For this, if σ is a path in X, we have to show that the two paths $g \circ \sigma$ and $\tau(f \circ \sigma)\tau^{-1}$ are homotope. For every s in Δ^1 consider the path $c_s : t \mapsto H(x, 1 - st)$ and let

$$H' : \Delta^1 \times \Delta^1 \to Y$$

be the map such that for every s in Δ^1, the map $H'(., s)$ corresponds to the loop $c_s^{-1} H(\sigma(.), 1 - s)c_s$. The map H' is continuous and we have the following relations

$$\begin{aligned} &H(., 0) = c_0^{-1} H(\sigma(.), 1)c_0 \simeq g \circ \sigma, \\ &H(., 1) = c_1^{-1} H(\sigma(.), 0)c_1 \simeq \tau(f \circ \sigma)\tau^{-1} \quad && \text{because } c_1 \text{ is nothing else than the path } \tau^{-1}, \\ &H(0, s) = H(1, s) = g(x) && \text{for every } s \text{ in } \Delta^1. \end{aligned}$$

In other words, we have defined our homotopy.

In the case where the homotopy H is an homotopy relatively to the point $\{x\}$, then $f(x) = g(x)$ and the path τ is nothing else than the identity path of $f(x)$. It follows then directly that u is the identity of $\pi_1(Y, f(x))$ and then that $f_* = g_*$. □

Corollary 24. *Let X and Y be two topological spaces and x a point in X.*

If $f : X \to Y$ is an equivalence of homotopy, then the map

$$f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$$

is a groups isomorphism.

Proof. Let choose $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to the identity of X and Y respectively. By applying the previous proposition to the homotopic maps $g \circ f$ and id_X, we deduce that $(g \circ f)_*$, which is isomorphic to $g_* \circ f_*$, is a groups isomorphism. The same is true for the map $f_* \circ g_*$ and thus is the map f_* both injective and surjective, hence an isomorphism. □

Corollary 25. *Let X and Y be two path-connected topological spaces with the same homotopy type.*

Then we have a groups isomorphism

$$\pi_1(X) \simeq \pi_1(Y).$$

Corollary 26. *The fundamental group of a contractible space is trivial.*

Exercise 12. What is the first fundamental group of the Möbius band defined by the quotient of the space $[0,1]\times[-1,1]$ by the relation $(0,s)\sim(1,-s)$?

Exercise 13. Show that the fundamental group of the complementary in $\mathbb{C}^2$ of two crossing lines is isomorphic to $\mathbb{Z}^2$.

Here a proposition which permits us to verify if a loop is homotopically trivial or not :

Proposition 27. *Let X be a topological space and x a point in X.*

Let σ a loop in X with endpoint x. We note by the isomorphism $\Delta^1/\{0,1\}\simeq\mathbb{S}^1$ that σ can be seen as a map $\alpha:\mathbb{S}^1\to X$ such that $\alpha(1)=x$.

Then σ is trivial in $\pi_1(X,x)$ if and only if the corresponding map α may be extended continuously to a map $\mathbb{B}^2\to X$.

Proof. First assume that the map α can be extended to a map β from the ball $\mathbb{B}^2$ to X. Let $i:\mathbb{S}^1\to\mathbb{B}^2$ denotes the canonical inclusion, we then have the following commutative diagram.

$$\begin{tikzcd} & \pi_1(\mathbb{B}^2,1) \arrow[d, "\beta_*"] \\ \pi_1(\mathbb{S}^1,1) \arrow[ur, "i_*"] \arrow[r, "\alpha_*"'] & \pi_1(X,x) \end{tikzcd}$$

But the canonical path $\tau:\Delta^1\to\Delta^1/\{0,1\}\simeq\mathbb{S}^1$ verifies $\alpha_*(\tau)=\sigma$. Hence $\alpha_*(\tau)$ must be trivial since $\mathbb{B}^2$ is contractible.

For the reverse, let H be an homotopy from σ to the constant path of value x. We see from the border conditions on H that H may be factorize to a continuous map $H_1:\mathbb{S}^1\times\Delta^1/\mathbb{S}^1\times\{1\}\to X$ with $H_1(.,0)=\alpha$. But $\mathbb{S}^1\times\Delta^1/\mathbb{S}^1\times\{1\}\simeq\mathbb{B}^2$ (to see this, we send the circle $\mathbb{S}^1$ at a height s to the circle of center 0 and radius $1-s$ in $\mathbb{B}^2$). With this identification, H_1 then defines our desire extension. □

For now, we know (even if we do not have prove it) that the fundamental group of the circle $\mathbb{S}^1$ is $\mathbb{Z}$. The next exercise shows that hopefully this fundamental group is not trivial :

Exercise 14. Show that if the circle $\mathbb{S}^1$ has a trivial fundamental group, then the fundamental group of every topological space is trivial.

Exercise 15. Let X be a path-connected topological space. Show that the following assertions are all equivalent :

1. every continuous map from $\mathbb{S}^1$ to X may be extended continuously on a map from $\mathbb{B}^2$ to X,
2. there exists a point x in X such that the group $\pi_1(X,x)$ is trivial,
3. for every point x in X, the group $\pi_1(X,x)$ is trivial,
4. for every points s and t in X, the set $\pi_1(X,s,t)$ is a singleton,

Definition 28. We say that a path-connected space is *simply connected* if it verifies one of the assertion of the previous exercise.

For example, a contractible space is simply connected.

Now we give a result in order to show how powerful the theory is.

Theorem 29 (Brouwer fixed point theorem)**.** *Let $n\geq 1$ be a positive integer and let $f:\mathbb{B}^n\to\mathbb{B}^n$ a continuous map.*

Then the map f has a fixed point.

The case $n=1$ is already known and we will just give here a proof in the case $n=2$, the general case requires more topological tools.

Proof. Let assume that there exists such a map f with no fixed point and consider the map $r : \mathbb{B}^2 \to \mathbb{S}^1$ defined as follows : for a point x in $\mathbb{B}^2$, then $r(x)$ correspond to the unique intersection point between the sphere $\mathbb{S}^1$ and the half-line with source (not-included) $f(x)$ which passes through x. It is possible to explicit the map r which shows that r is a continuous map (we left this verification to the reader).

Moreover, if i denotes the canonical inclusion from $\mathbb{S}^1$ to $\mathbb{B}^2$, then we have that $r \circ i$ is equal to the identity of $\mathbb{S}^1$. Then the composition

$$\pi_1(\mathbb{S}^1, x) \xrightarrow{i_*} \pi_1(\mathbb{B}^2, x) \xrightarrow{r_*} \pi_1(\mathbb{S}^1, x)$$

is also equal to the identity of $\pi_1(\mathbb{S}^1, x)$ (where x is an arbitrary point in $\mathbb{S}^1$).

But the fundamental group of $\mathbb{B}^2$ is trivial because $\mathbb{B}^2$ is simply connected and the fundamental group of the circle is non-trivial, this is then a contradiction. □

Proposition 30 (Van Kampen theorem (low version))**.** *Let X be a topological space which may be recovered by two open subsets U_1 and U_2 such that the intersection $U_0 = U_1 \cap U_2$ is non-empty. We also assume that U_0 is path-connected and let choose b an element of U_0.*

Denote by i_1 and i_2 the canonical inclusions of U_1 and U_2 in X respectively.

Then the group $\pi_1(X, b)$ is generated by $i_{1*}\pi_1(U_1, b)$ and $i_{2*}\pi_1(U_2, b)$.

Proof. Let σ be a loop in X with endpoint b. For each element t in Δ^1, there exists a connected open neighborhood V_t of t such that $\sigma(V_t)$ is contained in one of the U_i's. By compactness of Δ^1, there then exists a subdivision $0 = t_0 < t_1 < \cdots < t_n = 1$ of Δ^1 such that for each j, the image by σ of the interval $[t_j, t_{j+1}]$ is contained in one of the U_i's and $\sigma(t_j)$ is an element of U_0. By construction, the restriction of σ to $[t_j, t_{j+1}]$ may be view, after the choice of an homeomorphism $[t_j, t_{j+1}] \simeq \Delta^1$, as a path σ_j with source $\sigma(t_j)$ and target $\sigma(t_{j+1})$ contained in one of the U_i's.

For each $0 < j < n$, let choose a path τ_j in U_0 with source b and target $\sigma(t_j)$ (for $j = 0$ or $j = n$, we will also denote by τ_j the constant path of value b) and let α_j denotes the loop $\tau_{j+1}^{-1}\sigma_j\tau_j$ for $0 \leq j < n$.

Then σ is homotopic to the concatenate loop $\alpha_{n-1} \dots \alpha_1\alpha_0$ and by construction, each of the α_j defines an element of $i_{1*}\pi_1(U_1, b)$ or $i_{2*}\pi_1(U_2, b)$. □

Corollary 31. *We keep the same notations as the previous proposition and we assume moreover that U_1 and U_2 are simply-connected.*

Then X is simply connected.

Proposition 32. *Let $n \geq 2$ be a positive integer.*

Then $\mathbb{S}^n$ is simply connected.

Proof. Let N and S be two distinct points in $\mathbb{S}^n$ and denote by U_1 and U_2 the open subset $\mathbb{S}^n\backslash\{N\}$ and $\mathbb{S}^n\backslash\{P\}$ respectively. These subsets form a recovering of $\mathbb{S}^n$, are simply-connected (they are both homeomorphic to $\mathbb{R}^n$) and their intersection $U_1 \cap U_2$ is homeomorphic to $\mathbb{S}^{n-1}$ and thus is path-connected.

The previous corollary concludes. □

Exercise 16. Let m and n be two positive integers and let $C_1, \dots, C_n$ be open convex subsets of $\mathbb{R}^n$.

We assume that for all i, j, k, the intersections $C_i \cap C_j \cap C_k$ are non-empty.

Show that the union $\bigcup_i C_i$ is a simply-connected space.

Proposition 33. *Let n be a positive integer different from 2.*

Then $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^2$.

Proof. We already know the result in the case where $n = 1$ so we assume $n \geq 3$.

If such an isomorphism exists, then we should also have an isomorphism between $\mathbb{R}^n\backslash\{0\}$ and $\mathbb{R}^2\backslash\{0\}$. But these two spaces have the same homotopy type as $\mathbb{S}^{n-1}$ and $\mathbb{S}^1$ respectively and so do not have the same fundamental group, hence a contradiction. □

Exercise 17. Let $n \geq 1$ be a positive integer.

What is the fundamental group of $\mathbb{R}^n\backslash\{0\}$?

Exercise 18. Let m and n be two non negative integers.

What is the fundamental group of $\mathbb{R}^m \backslash \mathbb{R}^n$ when $m \geq n+2$? What if $m = n+1$?

Exercise 19. Let $n \geq 1$ be a positive integer and $f : \Delta^1 \to \mathbb{B}^n$ an injective map.

What is the fundamental group of the quotient space $\mathbb{B}^n / f(\Delta^1)$?

Exercise 20. Let m and n be two positive integers.

Show that the m-torus $\mathbb{T}^m$ is isomorphic to the n-sphere $\mathbb{S}^n$ if and only if $m = n = 1$.

Exercise 21.

1. Let $n \geq 2$ an integer. Show that there does not exist a non-empty open subset of $\mathbb{R}$ homeomorphic to an open subset of $\mathbb{R}^n$.
2. Let $n \geq 3$ an integer. Show that there does not exist a non-empty open subset of $\mathbb{R}^2$ homeomorphic to an open subset of $\mathbb{R}^n$.

The previous exercise is a special case of the „invariance of domain" :

Theorem 34. *Let n and m be two distinct positive integers.*

Then there does not exist a non-empty open subset of $\mathbb{R}^m$ homeomorphic to an open subset of $\mathbb{R}^n$.

4 Homotopy theory and covering spaces

We will continue our study of liftings for covering spaces. We have already seen a condition of unicity of a lifting but for now, we do not have treat the existence part.

Theorem 35 (Paths lifting theorem)**.** *Let $p : X \to B$ be a covering space and $\sigma : \Delta^1 \to X$ a path in X with source s.*

Let choose an arbitrary element x on the fibre $p^{-1}(s)$.

Then there exist a unique lifting $\overline{\sigma}$ of σ for p such that $\overline{\sigma}(0) = x$.

Proof. The unicity part follows from the unicity of lifting because Δ^1 is connected. We thus just have to deal with the existence.

Let define A to be the subset of Δ^1 of the element t such that the restriction of σ at $[0, t]$ admits a lifting with value x at 0.

The subset A contains 0 and thus is non-empty. Moreover, if t is in A, choose such a lifting $\overline{\sigma}$ of the restriction of σ at $[0, t]$, a trivial neighborhood U of $\sigma(t)$ such that we get an homeomorphism $p^{-1}(U) \simeq U \times F$ for a non-empty discrete topological space F and let f be the plate containing $\overline{\sigma}(t)$. Then there exists an open neighborhood of t in Δ^1 such that the image by σ is contained in U and it is then possible to prolongate the lifting $\overline{\sigma}$ in U by $\overline{\sigma}(u) = (\sigma(u), f)$ for every $u \in U$.

For every $t \in \Delta^1$, choose a trivialisable open neighborhood U_t of $\sigma(t)$ for p. The family $(\sigma^{-1}(U_t))_{t \in \Delta^1}$ forms an open recovering of the compact set Δ^1 and thus it is possible to find a finite subdivision $0 = t_0 < t_1 < \cdots < t_n = 1$ of Δ^1 such that for all i, the image of $[t_i, t_{i+1}]$ by σ is contained in a trivialisable subspace for p. It is then possible with the previous construction to prolongate our lifting by a finite number of steps and thus the result. ☐

A generalisation of the above theorem is the following

Theorem 36. *Let $n \geq 1$ be a positive integer, $p : X \to B$ a covering space and $f : (\Delta^1)^n \to X$ a continuous map.*

Let choose an arbitrary element x on $(\Delta^1)^n$ and an element y on the fibre $p^{-1}(f(x))$.

Then there exists a unique lifting $\overline{f}$ of f for p such that $\overline{f}(x) = y$.

We will not give the proof of this result since it is almost the same as the previous one but only with more technical issues. Note that the previous theorem corresponds to the special case $n = 1$ and $x = 0$.

The case $n = 2$ is also know as the „homotopy lifting theorem" and we will not in this note use more than these two versions.

We are now able to calculate the fundamental group of the sphere $\mathbb{S}^1$. For this, recall that the map $p : t \mapsto e^{2i\pi t}$ defines $\mathbb{R}$ as a covering space over $\mathbb{S}^1$. Let choose an element x in $\mathbb{S}^1$ and define a map from $\Pi_1(\mathbb{S}^1, x)$ to $\mathbb{Z}$ by sending a loop σ in $\mathbb{S}^1$ with endpoint x to the value $\overline{\sigma}(1) - \overline{\sigma}(0)$ where $\overline{\sigma}$ is a lifting from σ for p.

Note first that this map does not depend on a choice of a litfting of σ. Indeed, if we choose two elements t_0 and t_1 in the fibre $p^{-1}(\{x\})$ then there exists an integer k such that $t_1 = t_0 + k$. But now, if $\overline{\sigma}_0$ and $\overline{\sigma}_1$ are the lifting associated to σ for p with source t_0 and t_1 respectively, then the map $t \mapsto \overline{\sigma}_0(t) + k$ also defines a lifting of σ for p with source t_1 so is equal to $\overline{\sigma}_1$ by unicity, hence $\overline{\sigma}_0(1) - \overline{\sigma}_0(0) = \overline{\sigma}_1(1) - \overline{\sigma}_1(0)$.

Now, if H is an homotopy between two elements σ and τ in $\Pi_1(\mathbb{S}^1, x)$, let choose an arbitrary element t in the fibre $p^{-1}(\{x\})$ and consider the lifting $\overline{H}$ of H for p such that $H(0,0) = 1$. Then $\overline{H}(.,0)$ and $\overline{H}(.,1)$ correspond to a lifting of σ and τ respectively, and because the maps $\overline{H}(0,.)$ and $\overline{H}(1,0)$ are constant maps, this shows that the two liftings have the same endpoints. In other words, our previous map induces a map

$$\alpha : \pi_1(\mathbb{S}^1, x) \to \mathbb{Z}.$$

Moreover, it is not difficult to see that the previous map is a groups homomorphism.

- The map α is surjective : let n be an integer and consider the loop in $\mathbb{S}^1$ with endpoint x given by the association $t \mapsto e^{2i\pi tn}x$. Then $t \mapsto t_0 + tn$, where $t_0 \in p^{-1}(\{x\})$, is of course a lifting of our loop for p and so is n in the image of α.

- The map α is injective : let σ be an element of $\Pi_1(\mathbb{S}^1, x)$ and consider as before t_0 in the fibre $p^{-1}(\{x\})$ and $\overline{\sigma}$ the lifting with source t_0 of σ for p. Then if the homotopy class of σ is sent to 0 by α, this means that $\overline{\sigma}$ is a loop in $\mathbb{R}$, which is a simply connected space. In particular, there exists an homotopy $\overline{H}$ between $\overline{\sigma}$ and the constant loop of value t_0. Bur then, the map $p \circ \overline{H}$ defines an homotopy between σ and the constant loop of value x and thus is σ the identity in $\pi_1(\mathbb{S}^1, x)$.

From what precede, we have (finally) a proof of our admitted result :

Theorem 37. *The fundamental group of the circle $\mathbb{S}^1$ is $\mathbb{Z}$.*

We will now give other consequences of the lifting theorems.

Proposition 38. *Let $p : X \to B$ be a covering space, x a point in X and b its image by p.*

Then the groups homomorphism

$$p_* : \pi_1(X, x) \to \pi_1(B, b)$$

is injective.

Proof. Let σ be a loop in X with basepoint x such that there exists an homotopy H between $p_*\sigma$ and the identity of b. By the previous theorem, let $\overline{H}$ be the unique lifting of H for p such that $\overline{H}(0,0) = x$.

The path $\overline{H}(.,0)$ corresponds to a lifting of $p_*\sigma$ with $\overline{H}(0,0) = x = \sigma(0)$. By unicity of the lifting, we then have that $\overline{H}(.,0) = \sigma$. The same argument applied for each side of the square $\Delta^1 \times \Delta^1$ shows that $\overline{H}$ is an homotopy between σ and the identity of x, the result then follows. □

Proposition 39. *Let $p : X \to B$ be a covering space. Let x and b be two elements on X and B respectively such that $p(x) = b$.*

Then for every t in the path-connected component of b in B, we get a well defined map

$$\pi_1(X, b, t) \to p^{-1}(t)$$

by sending the class of a path σ to the target of $\overline{\sigma}$ where $\overline{\sigma}$ is the unique lifting of σ for p with source x.

In particular, we have a group action of $\pi_1(X, b)$ on the space $p^{-1}(b)$ called the monodromy action.

Proof. We get of course by the „paths lifting theorem" a map

$$\Pi_1(X, b, t) \to p^{-1}(t).$$

The fact that this map induces a map from $\pi_1(X, b, t)$ comes from the „homotopy lifting theorem".

Finally, it is not a difficult exercise to check the axioms of a group action. $\square$

Proposition 40. *Let $p : X \to B$ be a covering space, b a point in B and x an element of the fibre $F := p^{-1}(\{b\})$.*

Then the stabilizer subgroup with respect to x for the monodromy action of $\pi_1(B, b)$ on F is equal to $p_*\pi_1(X, x)$.

Proof. Let σ be an element of this stabilizer and let $\overline{\sigma}$ the unique lifting for p with source x. The fact that σ belongs to the stabilizer with respect to x means that $\overline{\sigma}$ is a loop in X with endpoint x and we have by construction that $p_*\overline{\sigma} = \sigma$.

Reciprocally, let $\sigma = p_*\tau$ an element of $p_*\pi_1(X, x)$. Then τ is nothing else than the lifting of σ for p with source x. The fact that τ is a loop shows that σ stabilize x. $\square$

Proposition 41. *We keep the notations of the previous proposition and we assume that B is path-connected.*

Then the map which sends an element of F to its path-connected component induces a bijection

$$\pi_1(B, b)\backslash F \to \pi_0(X)$$

between the orbits of $\pi_1(B, b)$ in F and the set $\pi_0(X)$ of the path-connected components of X.

In particular, X is path-connected if and only if $\pi_1(B, b)$ acts transitively on F.

Proof.

- The map $F \to \pi_0(X)$ is surjective : let X' be a path-connected component of X and x' an arbitrary element of X'. Choose a path σ in B from source $p(x')$ and target b and consider the unique lifting $\overline{\sigma}$ of σ for p with source x'. Then the target of $\overline{\sigma}$ defines an element of $F \cap X'$.
- Two elements in the same orbit of $\pi_1(B, b)$ in F are by definition linked by a path and thus are in the same path-component of X. We thus obtain an induces map

$$\pi_1(B, b)\backslash F \to \pi_0(X).$$

- Finally, let f and f' two elements of F lying in the same path-component of X and choose a path σ between f and f'. Then $p \circ \sigma$ defines a loop in B with endpoint b and by construction, we have $(p \circ \sigma).f = f'$.

$\square$

Corollary 42. *Let $p : X \to B$ be a covering, b an element in B and x an element of the fibre $F := p^{-1}(\{b\})$.*

Assume that X is path-connected. Then the monodromy action of $\pi_1(B, b)$ on F induces a bijection

$$p_*\pi_1(X, x)\backslash\pi_1(B, b) \simeq F.$$

Corollary 43. *We keep the same assumptions as the previous corollary.*

Then p is an homeomorphism if and only if the groups homomorphism $p_* : \pi_1(X, x) \to \pi_1(B, b)$ is surjective.

Proof. Since the direct sense is clear, we assume that the morphism p_* is surjective (and hence an isomorphism because p is a covering space). By the previous corollary, the fibre F of p over b is a singleton and because B is connected, all fibres of p are singleton and then is p an homeomorphism. $\square$

Corollary 44. *Let $p : X \to B$ a covering space with X path-connected and B simply connected.*

Then p is an homeomorphism.

Exercise 22. Let X be a path-connected topological space.

Show that if X is simply connected then every covering space of X is trivial.

Proposition 45. *Let G be a group which acts evenly on a path-connected topological space X and denote by $p : X \to B := G\backslash X$ the associated covering space. Let choose b an element in B and x in the fibre $p^{-1}(\{b\})$.*

Then for all element σ of $\pi_1(B,b)$, there exists an unique element g in G such that $\sigma.x = g^{-1}.x$.

Moreover, the map so defined

$$\alpha : \pi_1(B,b) \to G$$

is a groups homomorphism and induces an isomorphism

$$\pi_1(B,b)/p_*\pi_1(X,x) \to G.$$

Proof. The fact that g exists and is unique is clear.

We will now see that the map α is a groups homomorphism : let σ and τ be two elements in $\Pi_1(B,b)$ and consider $\overline{\sigma}$ and $\overline{\tau}$ the liftings respectively of σ and τ for p with source x. Then $\alpha(\sigma)^{-1}.\tau$ is the lifting of τ with source $\alpha(\sigma)^{-1}.x = \overline{\sigma}(1)$ and thus $(\tau\sigma).x$ is nothing else than the target of the path $\alpha(\sigma)^{-1}.\tau$ which is $\alpha(\sigma)^{-1}.(\alpha(\tau)^{-1}.x) = (\alpha(\tau)\alpha(\sigma))^{-1}.x$.

Moreover, the map α is surjective because $\pi_1(B,b)$ acts transitively on the fibre $p^{-1}(\{b\})$ (by path-connectedness of X) and the kernel is exactly the stabilizer with respect to x for the monodromy action of $\pi_1(B,b)$ on $p^{-1}(\{b\})$ and thus is equal to $p_*\pi_1(X,x)$. □

Corollary 46. *Let G be a group which acts evenly on a simply connected topological space X.*

Then we have an isomorphism

$$\pi_1(G\backslash X) \simeq G.$$

Examples.

1. We already have see the covering space $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ and the isomorphism $\mathbb{R}/\mathbb{Z} \simeq \mathbb{S}^1$ show that $\pi_1(\mathbb{S}^1) = \mathbb{Z}$.

2. More generally, the covering space $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ and the isomorphism $\mathbb{R}^n/\mathbb{Z}^n \simeq \mathbb{T}^n =: (\mathbb{S}^1)^n$ show that $\pi_1(\mathbb{T}^n) = \mathbb{Z}^n$.

3. Let $n \geq 2$ be a positive integer. By the antipodal action $x \mapsto -x$ on the sphere $\mathbb{S}^n$, we get a covering space $\mathbb{S}^n \to (\mathbb{Z}/2\mathbb{Z})\backslash\mathbb{S}^n \simeq \mathbb{RP}^n$. The fact that $\mathbb{S}^n$ is simply connected for $n \geq 2$ then shows that $\pi_1(\mathbb{RP}^n) = \mathbb{Z}/2\mathbb{Z}$.

   In the case $n = 1$, we have $\pi_1(\mathbb{RP}^1) = \mathbb{Z}$ because of the homeomorphism $\mathbb{RP}^1 \simeq \mathbb{S}^1$.

Exercise 23. Let $n \geq 0$ and $p \geq 1$ be two integers and denote by $\mathbb{U}_p$ the subset of $\mathbb{C}$ of the p-roots of the unity.

Then $\mathbb{U}_p$ acts on the sphere

$$\mathbb{S}^{2n+1} \simeq \{(z_0, \ldots, z_n), |z_0|^2 + \cdots + |z_n|^2 = 1\}$$

by

$$\varphi.(z_0, \ldots, z_n) = (\varphi z_0, \ldots, \varphi z_n)$$

and we define the *Lens space* $L(n,p)$ as the quotient $\mathbb{U}_p\backslash\mathbb{S}^{2n+1}$.

Show that the canonical map $\mathbb{S}^{2n+1} \to L(n,p)$ is a covering space of degree p.

What is the fundamental group of $L(n,p)$?

Exercise 24. Let G be the group of the homeomorphisms of the plane $\mathbb{R}^2$ generates by the two elements

$$t : (x,y) \mapsto (x+1, y) \text{ and } s : (x,y) \mapsto (-x, y+1).$$

This group acts on $\mathbb{R}^2$ and we define the *Klein bottle* K as the quotient space $G\backslash\mathbb{R}^2$.

1. Show that K is also homeomorphic to the quotient of the square $\Delta^1 \times \Delta^1$ by the relations $(0,y) \sim (1,y)$ and $(x,0) \sim (1-x,1)$.

2. Show that the fundamental group of K is isomorphic to a group with two generators s and t and the relation $tst = s$.
3. If A is a group which acts evenly on a topological space X and if H is a subgroup of G, show that H also acts evenly on X and that the canonical map
$$H\backslash X \to G\backslash X$$
is a covering space.
4. By considering the group of the homeomorphisms of the plane $\mathbb{R}^2$ generates by the two elements
$$t : (x, y) \mapsto (x + 1, y) \text{ and } u : (x, y) \mapsto (x, y + 2)$$
find a covering of the Klein bottle K with the 2-torus $\mathbb{T}^2$ and explicit the associated map
$$\pi_1(\mathbb{T}^2) \to \pi_1(K).$$

Exercise 25. Let G be the subgroup of the isometries of $\mathbb{R}^2 \simeq \mathbb{C}$ generates by the conjugation $z \mapsto \overline{z}$ and the multiplication by $e^{\frac{i\pi}{5}}$.

If we identify $\mathbb{S}^3$ as followed
$$\mathbb{S}^3 := \{(z_1, z_2) \in \mathbb{C}^2, |z_1|^2 + |z_2|^2 = 1\}$$
then we have a group action of G on $\mathbb{S}^3 \times \mathbb{S}^2$ given by
$$g.(z_1, z_2, \rho) := (gz_1, gz_2, \varepsilon(g)\rho)$$
where $\varepsilon(g) = 1$ if g is a rotation and -1 otherwise.

Show that G is a finite group, and that we get a covering space
$$p : \mathbb{S}^3 \times \mathbb{S}^2 \to G\backslash\mathbb{S}^3 \times \mathbb{S}^2.$$

What is the fundamental group of the quotient $G\backslash\mathbb{S}^3 \times \mathbb{S}^2$?

Theorem 47 (Borsuk-Ulam theorem)**.** *Let $n \geq 1$ be a positive integer.*

Then there does not exist a continuous map $f : \mathbb{S}^n \to \mathbb{S}^{n-1}$ such that for all $x \in \mathbb{S}^n$, we have that
$$f(-x) = -f(x).$$

The case $n = 1$ is easy and we will here just give a proof in the case $n = 2$, the general case uses more involved topological techniques.

Proof. By considering the antipodal action $x \mapsto -x$ on $\mathbb{S}^1$ and $\mathbb{S}^2$ respectively, we obtain two covering spaces $p_1 : \mathbb{S}^1 \to (\mathbb{Z}/2\mathbb{Z})\backslash\mathbb{S}^1 \simeq \mathbb{S}^1$ and $p_2 : \mathbb{S}^2 \to (\mathbb{Z}/2\mathbb{Z})\backslash\mathbb{S}^2 \simeq \mathbb{RP}^2$ and the hypothesis on f show that there exists a continuous map $g : \mathbb{RP}^2 \to \mathbb{S}^1$ such that the following diagram commutes.
$$\begin{tikzcd}
\mathbb{S}^2 \arrow[r, "f"] \arrow[d, "p_2"'] & \mathbb{S}^1 \arrow[d, "p_1"] \\
\mathbb{RP}^2 \arrow[r, "g"] & \mathbb{S}^1
\end{tikzcd}$$
This diagram induces a commutative square of groups homomorphisms
$$\begin{tikzcd}
\pi_1(\mathbb{S}^2, x) \arrow[r, "f_*"] \arrow[d, "p_{2*}"'] & \pi_1(\mathbb{S}^1, y) \arrow[d, "p_{1*}"] \\
\pi_1(\mathbb{RP}^2, z) \arrow[r, "g_*"] & \pi_1(\mathbb{S}^1, t)
\end{tikzcd}$$

where x is an arbitrary point in $\mathbb{S}^2$ and y, z and t the image of x by f, p_2 and $g \circ p_2 = p_1 \circ f$ respectively.

Let choose a path σ in $\mathbb{S}^2$ with source x and target $-x$. By assumption on f, the path $f_*(\sigma)$ in $\mathbb{S}^1$ has different source and target, namely y and $-y$, and thus the loop $p_{1*}f_*(\sigma)$ is not trivial in $\pi_1(\mathbb{S}^1, t)$, otherwise, if you choose an homotopy between $p_{1*}f_*(\sigma)$ and the constant loop with value t, then the unique lifting $\overline{H}$ of H for p_1 such that $\overline{H}(0,0) = y$ will be an homotopy between $f_*(\sigma)$ and the constant path of value y and thus we have $y = -y$ which is wrong.

But g_* must be trivial, indeed g_* is a groups homomorphism between $\pi_1(\mathbb{RP}^2, z) \simeq \mathbb{Z}/2\mathbb{Z}$ and $\pi_1(\mathbb{S}^1, t) \simeq \mathbb{Z}$. Then the relation $g_* \circ p_{2*} = p_{1*} \circ f_*$ shows that the loop $p_{1*}f_*(\sigma)$ must be trivial as well, which contradicts what precede. □

Now we give some consequences of this theorem :

Corollary 48. *Let $f : \mathbb{S}^n \to \mathbb{R}^n$ be a continuous map such that for all $x \in \mathbb{S}^n$, we have that*

$$f(-x) = -f(x).$$

Then there exists an element x in $\mathbb{S}^n$ such that $f(x) = 0$.

Proof. Otherwise, the map $g : x \mapsto \frac{f(x)}{|f(x)|}$ from $\mathbb{S}^n$ to $\mathbb{S}^{n-1}$ will contradict the Borsuk-Ulam theorem. □

Corollary 49. *Let $f : \mathbb{S}^n \to \mathbb{R}^n$ be a continuous map.*

Then there exists an element x in $\mathbb{S}^n$ such that $f(x) = f(-x)$. In particular, the map f is not injective.

Proof. The map $g : x \mapsto f(x) - f(-x)$ from $\mathbb{S}^n$ to $\mathbb{R}^n$ satisfies the hypothesis of the previous result and then has to vanish at some point. □

Corollary 50. *There does not exist a subset of $\mathbb{R}^n$ homeomorphic to $\mathbb{S}^n$.*

Exercise 26. Let $\{F_1, F_2, \ldots, F_{n+1}\}$ be a recovering of the n-sphere $\mathbb{S}^n$ with closed subsets.

Show that there exists an element x in $\mathbb{S}^n$ such that x and $-x$ are on a same F_i for a certain i.

Exercise 27. Prove the general lifting theorem :

Let $p : X \to B$ be a covering space with B path-connected and let $f : B' \to B$ be a continuous map.

Let b' an element in B', define $b = f(b')$ and let choose an element x in the fibre $p^{-1}(\{b\})$.

Then there exists a lifting F of f for p such that $F(b') = x$ if and only if we have an inclusion

$$f_*\pi_1(B', b') \subset p_*\pi_1(X, x).$$

In particular, if B' is simply-connected, then a lifting of f always exists.

Just give some more results of existence without proofs :

Definition 51. We say that a covering $p : X \to B$ with X connected is *universal* if for all covering space $p' : X' \to B$ with X' connected, for all x and x' in X and X' respectively such that $p(x) = p(x')$ then there exists a unique morphism of covering spaces $f : p \to p'$ such that $f(x) = x'$.

In other words, a covering space of B is universal if he may be factorize by all the other coverings of B.

Proposition 52. *An universal covering space is always Galois.*

Proposition 53. *If $p : X \to$ is a covering space with X simply connected, then p is an universal covering space.*

Theorem 54. *Let B be an Hausdorff topological space which is path-connected and semi-locally simply connected (that mean that for every b in B, there exists an open neighborhood U of b in B such that the image of $\pi_1(U, b)$ in $\pi_1(X, b)$ is trivial).*

Then B admits a simply-connected (and hence universal) covering space.

If we combine the existence of an universal covering space with the corollary 46, we then see that if we describe a topological space as a quotient of an even action of a group G on its universal covering, then we have that its fundamental group is isomorphic to G. That is what we have done during the examples following the corollary 46.

5 Bonus

We will give here an other point of view of composition of paths and homotopy in a more theoretical level. The aim of this section is just to shows to the reader that it is possible to define these notions without using any element (we define a map $f : \Delta^1 \to X$ for example, but we never give the concrete value $f(t)$ for a certain t in Δ^1). It is this point of view which may be generalize in other contexts (for example, see what we call an ∞-category). We will not go far in this theory and some details or precisions will be omitted, the objective of this section is just to underline the fact that if the readers deal with axioms which are closely related to the following results, then it will be good for him to realize that on the category of topological spaces, he is just doing (more or less) homotopy.

For this, we will note by Δ^2 the subspace of $\mathbb{R}^2$ defines as follows :

$$\Delta^2 := \{(x, y) \in \mathbb{R}^2, x \geq 0, y \geq 0, x + y \leq 1\}.$$

The neighborhood of Δ^2 is then the following union

$$\partial\Delta^2 = A_0 \cup A_1 \cup A_2$$

where

$$\begin{aligned} &A_0 := \{(x, y) \in \mathbb{R}^2, x \geq 0, y \geq 0, x + y = 1\}, \\ &A_1 := \{(x, 0), 0 \leq x \leq 1\} \text{ and} \\ &A_2 := \{(0, y) \in \mathbb{R}^2, 0 \leq y \leq 1\}. \end{aligned}$$

Note that each of the $A_i's$ can be identified with Δ^1 and thus we define three maps

$$d_i^2 : \Delta^1 \to A_i \to \partial\Delta^2.$$

Remark. We will sometimes write the composition

$$\Delta^1 \xrightarrow{d_i^2} \partial\Delta^2 \longrightarrow \Delta^2$$

also by d_i^2.

Definition 55. A *triangle in* X is a map from $\partial\Delta^2$ to X.

With the description of $\partial\Delta^2$ with the $A_i's$ describe before, we see that it is the same things as the data of three paths f, g, h in X such that

- sources of f and h coincide,
- the target of f is the source of g,
- targets of g and h coincide.

We will then write (f, g, h) for such a triangle in X.

Definition 56. Let (f, g, h) be a triangle in X.

We say that this triangle *commutes* if there exists a map $u : \Delta^2 \to X$ such that the following diagram commutes (where the map from $\partial\Delta^2$ to Δ^2 denote the canonical inclusion).

$$\begin{tikzcd} \partial\Delta^2 \arrow[r, "(f,g,h)"] \arrow[d] & X \\ \Delta^2 \arrow[ur, "u"'] & \end{tikzcd}$$

Definition 57. Let f and g be two paths in X such that the target of f coincide with the source of g.

A *composition of f by g in* X is a path h in X such that (f, g, h) defines a commutative triangle in X.

Remark. We will explain a little more why we use this definition of a composition. We have defined in section 3 a composition of paths in a topological space X, but if we do homotopy, a composition like this is not really relevant, we just have to know how to compose *up to homotopy*. That is, if we have f and g two paths in X where the source of f coincide with the target of g and if we denote by s and t respectively the source of f and the target of g, then we just have to know how to define a path h with source s and target t *homotopic* to our gf, or in other words we just need to find a path such that $h^{-1}gf$ is a loop homotopic to the identity of s. We will now use our proposition 27, a loop in X, view as a map $\mathbb{S}^1 \to X$, is homotopically trivial if and only if he may be extended on a map $\mathbb{B}^2 \to X$. With the isomorphisms $\partial\Delta^2 \simeq \mathbb{S}^1$ and $\Delta^2 \simeq \mathbb{B}^2$, we know see our link with the previous definition.

If x is a point in X, define 1_x to be the constant path in X with value x. We then define five relations on $\Pi_1(X, s, t)$ in the following way :

- $f \sim_1 g$ if g is a composition of f by 1_t in X,
- $f \sim_2 g$ if g is a composition of 1_s by f in X,
- $f \sim_3 g$ if f is a composition of g by 1_t in X,
- $f \sim_4 g$ if f is a composition of 1_s by g in X,
- $f \sim_5 g$ if there exists a map $h : \Delta^1 \times \Delta^1 \to X$ such that $h(.,0) = f$, $h(.,1) = g$, $h(0,1) = 1_s$, $h(1,.) = 1_t$

Note that $\sim_5$ corresponds to the homotopy equivalence of paths defined in section 3..

Lemma 58. *The five relations describe before are equal. We will then denote this relation simply by* $\sim$*.*

Consequence 59. *The relation $\sim$ is an equivalence relation in the set of paths in X with source s and target t and we define $\pi_1(X, s, t)$ the quotient of this previous set by the relation $\sim$.*

Remark. Of course, the π_1 that we construct here is the same as the fundamental group defined in section 3.

We will end this section here. As says before the aim is just to underline how the theory may be view in a more abstract language. And indeed we see that we almost only use

- the data of objects Δ^0, Δ^1 and Δ^2 with maps between them (the source and target map $s, t : \Delta^0 \to \Delta^1$, a retraction map $\Delta^1 \to \Delta^0$ and the $d_2^i : \Delta^1 \to \Delta^2$),
- an object X where maps $\Delta^j \to X$ have a sense

and then it is possible to copy a theory of homotopy on X and to define for example a fundamental group π_1.

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Exercises

We recall here the different exercises.

Exercise 1. Let $p : X \to B$ and $p' : X' \to B'$ be two covering spaces.

Show that the map from $X \times X'$ to $B \times B'$ which sends (x, x') to $(p(x), p'(x'))$ is a covering space of $B \times B'$.

Exercise 2. Let $p : X \to B$ be a covering space with B connected.

Show that all the fibers are homeomorphic to a same non-empty discrete topological space.

Exercise 3. Let $p : Y \to X$ and $q : X \to B$ be two covering spaces and assume that the covering q is finite.

1. Show that the composition map $q \circ p : Y \to B$ is a covering space of B. We will denote this covering as $q_!(p)$.

2. If moreover p and q are finite covering of degree d and d' respectively, what is the degree of $q_!(p)$?

3. (this question uses the language of category theory)

   Show that $q_! : Cov(X) \to Cov(B)$ is a functor. Moreover, if $q' : X' \to X$ is another finite covering, show that we get a natural transformation

$$(q \circ q')_! \simeq q_! \circ q'_!.$$

Exercise 4. Let $p : X \to B$ be a covering space and $f : B' \to B$ a continuous map between topological spaces.

Define

$$X \times_B B' := \{(x, b') \in X \times B', p(x) = f(b')\}$$

and denote by q the map from $X \times_B B'$ to B' which sends a couple (x, b') to b'.

1. Show that q is a covering space of B'. It is call the *base change of p by f* or the *pull-back of p by f* and is often written as $f^*(p)$.

2. If moreover p is a finite covering of degree d, what is the degree of $f^*(p)$?

3. (this question uses the language of category theory)

   Show that $f^* : Cov(B) \to Cov(B')$ is a functor. Moreover, if $f' : B'' \to B'$ is another continuous map between topological spaces, show that we get a natural transformation

$$(f \circ f')^* \simeq f'^* \circ f^*.$$

Exercise 5. (this exercise uses the language of category theory)

We use the notations and results of exercices 3. and 4..

Show that if $f : B' \to B$ is a finite covering of B, then the functor $f_!$ is left adjoin to the functor f^*.

Exercise 6. Let $p : X \to B$ be a covering space and $f : B' \to B$ a continuous map.

Show that we have a bijection between the liftings of f for p and the section of the pull-back covering $f^*(p)$ defined in exercise 4.

Exercise 7. Let $p : X \to B$ be a covering space with Y connected.

Show that p is a Galois covering if and only if the group $Aut(p)$ acts transitively on each fibres of p.

Exercise 8. Let B and B' be two topological spaces and b (respectively b') a point in B (respectively B').

Show that the canonical map

$$\pi_1(B \times B', (b, b')) \to \pi_1(B, b) \times \pi_1(B', b')$$

defined by the projections maps $B \times B' \to B$ and $B \times B' \to B'$ is an isomorphism.

In particular, the fundamental group of the n-torus $\mathbb{T}^n := (\mathbb{S}^1)^n$ is $\mathbb{Z}^n$.

Exercise 9. Let X be a topological space and define CX by the quotient of $X \times \Delta^1$ by the relation $(x, 1) \sim (y, 1)$ for every elements x and y in X.

Show that X is a contractible space.

Exercise 10. Show that the fundamental group of a contractible topological space is trivial.

Exercise 11. Verify the following results :

1. a contractible space has the same homotopy type as the space Δ^0,
2. if $n \geq 1$ is a positive integer, then the n-sphere $\mathbb{S}^n$ has the same homotopy type as $\mathbb{R}^{n+1} \backslash \{0\}$,
3. if X and Y are two topological spaces with Y contractible and if y is an arbitrary element of Y, then $X \times Y$ has the same homotopy type as $X \times \{y\}$,
4. let define the Möbius band M as the quotient of the space $[0, 1] \times [-1, 1]$ by the relation $(0, s) \sim (1, -s)$. Then M has the same homotopy type as the circle $\mathbb{S}^1$.

Exercise 12. What is the first fundamental group of the Möbius band defined by the quotient of the space $[0, 1] \times [-1, 1]$ by the relation $(0, s) \sim (1, -s)$?

Exercise 13. Show that the fundamental group of the complementary in $\mathbb{C}^2$ of two crossing lines is isomorphic to $\mathbb{Z}^2$.

Exercise 14. Show that if the circle $\mathbb{S}^1$ has a trivial fundamental group, then the fundamental group of every topological space is trivial.

Exercise 15. Let X be a path-connected topological space. Show that the following assertions are all equivalent :

1. every continuous map from $\mathbb{S}^1$ to X may be extended continuously on a map from $\mathbb{B}^2$ to X,
2. there exists a point x in X such that the group $\pi_1(X, x)$ is trivial,
3. for every point x in X, the group $\pi_1(X, x)$ is trivial,
4. for every points s and t in X, the set $\pi_1(X, s, t)$ is a singleton,

Exercise 16. Let m and n be two positive integers and let $C_1, \ldots, C_n$ be open convex subsets of $\mathbb{R}^n$.

We assume that for all i, j, k, the intersections $C_i \cap C_j \cap C_k$ are non-empty.

Show that the union $\bigcup_i C_i$ is a simply-connected space.

Exercise 17. Let $n \geq 1$ be a positive integer.

What is the fundamental group of $\mathbb{R}^n \backslash \{0\}$?

Exercise 18. Let m and n be two non negative integers.

What is the fundamental group of $\mathbb{R}^m \backslash \mathbb{R}^n$ when $m \geq n + 2$? What if $m = n + 1$?

Exercise 19. Let $n \geq 1$ be a positive integer and $f : \Delta^1 \to \mathbb{B}^n$ an injective map.

What is the fundamental group of the quotient space $\mathbb{B}^n / f(\Delta^1)$?

Exercise 20. Let m and n be two positive integers.

Show that the m-torus $\mathbb{T}^m$ is isomorphic to the n-sphere $\mathbb{S}^n$ if and only if $m = n = 1$.

Exercise 21. 1. Let $n \geq 2$ an integer. Show that there does not exist a non-empty open subset of $\mathbb{R}$ homeomorphic to an open subset of $\mathbb{R}^n$.

2. Let $n \geq 3$ an integer. Show that there does not exist a non-empty open subset of $\mathbb{R}^2$ homeomorphic to an open subset of $\mathbb{R}^n$.

Exercise 22. Let X be a path-connected topological space.

Show that if X is simply connected then every covering space of X is trivial.

Exercise 23. Let $n \geq 0$ and $p \geq 1$ be two integers and denote by $\mathbb{U}_p$ the subset of $\mathbb{C}$ of the p-roots of the unity.

Then $\mathbb{U}_p$ acts on the sphere

$$\mathbb{S}^{2n+1} \simeq \{(z_0, \ldots, z_n), |z_0|^2 + \cdots + |z_n|^2 = 1\}$$

by

$$\varphi.(z_0, \ldots, z_n) = (\varphi z_0, \ldots, \varphi z_n)$$

and we define the *Lens space* $L(n, p)$ as the quotient $\mathbb{U}_p \backslash \mathbb{S}^{2n+1}$.

Show that the canonical map $\mathbb{S}^{2n+1} \to L(n, p)$ is a covering space of degree p.

What is the fundamental group of $L(n, p)$?

Exercise 24. Let G be the group of the homeomorphisms of the plane $\mathbb{R}^2$ generates by the two elements

$$t : (x, y) \mapsto (x + 1, y) \text{ and } s : (x, y) \mapsto (-x, y + 1).$$

This group acts on $\mathbb{R}^2$ and we define the *Klein bottle* K as the quotient space $G\backslash\mathbb{R}^2$.

1. Show that K is also homeomorphic to the quotient of the square $\Delta^1 \times \Delta^1$ by the relations $(0, y) \sim (1, y)$ and $(x, 0) \sim (1 - x, 1)$.

2. Show that the fundamental group of K is isomorphic to a group with two generators s and t and the relation $tst = s$.

3. If A is a group which acts evenly on a topological space X and if H is a subgroup of G, show that H also acts evenly on X and that the canonical map

$$H\backslash X \to G\backslash X$$

is a covering space.

4. By considering the group of the homeomorphisms of the plane $\mathbb{R}^2$ generates by the two elements

$$t : (x, y) \mapsto (x + 1, y) \text{ and } u : (x, y) \mapsto (x, y + 2)$$

find a covering of the Klein bottle K with the 2-torus $\mathbb{T}^2$ and explicit the associated map

$$\pi_1(\mathbb{T}^2) \to \pi_1(K).$$

Exercise 25. Let G be the subgroup of the isometries of $\mathbb{R}^2 \simeq \mathbb{C}$ generates by the conjugation $z \mapsto \overline{z}$ and the multiplication by $e^{\frac{i\pi}{5}}$.

If we identify $\mathbb{S}^3$ as followed

$$\mathbb{S}^3 := \{(z_1, z_2) \in \mathbb{C}^2, |z_1|^2 + |z_2|^2 = 1\}$$

then we have a group action of G on $\mathbb{S}^3 \times \mathbb{S}^2$ given by

$$g.(z_1, z_2, \rho) := (gz_1, gz_2, \varepsilon(g)\rho)$$

where $\varepsilon(g) = 1$ if g is a rotation and -1 otherwise.

Show that G is a finite group, and that we get a covering space

$$p : \mathbb{S}^3 \times \mathbb{S}^2 \to G\backslash\mathbb{S}^3 \times \mathbb{S}^2.$$

What is the fundamental group of the quotient $G\backslash\mathbb{S}^3 \times \mathbb{S}^2$?

Exercise 26. Let $\{F_1, F_2, \ldots, F_{n+1}\}$ be a recovering of the n-sphere $\mathbb{S}^n$ with closed subsets.

Show that there exists an element x in $\mathbb{S}^n$ such that x and $-x$ are on a same F_i for a certain i.

Exercise 27. Prove the general lifting theorem :

Let $p : X \to B$ be a covering space with B path-connected and let $f : B' \to B$ be a continuous map.

Let b' an element in B', define $b = f(b')$ and let choose an element x in the fibre $p^{-1}(\{b\})$.

Then there exists a lifting F of f for p such that $F(b') = x$ if and only if we have an inclusion

$$f_*\pi_1(B', b') \subset p_*\pi_1(X, x).$$

In particular, if B' is simply-connected, then a lifting of f always exists.