Winter school in Prishtina
Introduction to graph theory

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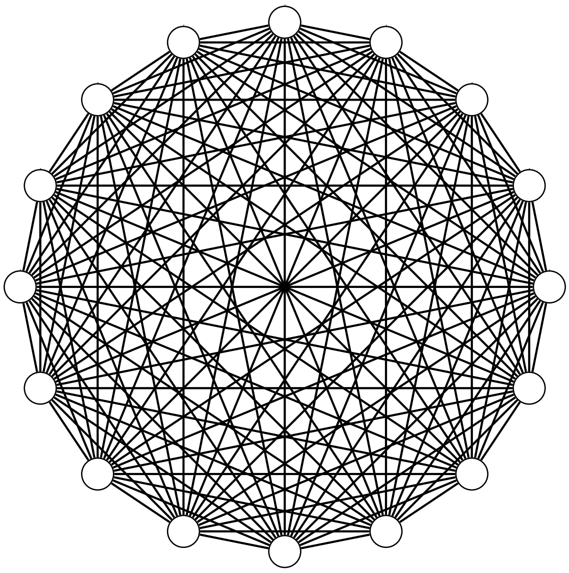

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Chapter 1

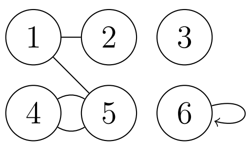
Introduction and preliminary definitions

1.1 Graphs and subgraphs

Definition 1.1. A *graph* is a pair of sets (V, E), where V is a set of *vertices* and E is a set of unordered pairs of elements of V called *edges*.

If v_1 and v_2 are two vertices, we denote by (v_1, v_2) or (v_2, v_1) the edge from v_1 to v_2.

Example 1.2 Let $V = (1, 2, 3, 4, 5, 6)$ and $E = \{(1, 2), (1, 5), (4, 5), (4, 5), (6, 6)\}$.



Definition 1.3. Let $G = (V, E)$ be a graph. Two vertices $v, w \in V$ are *adjacent*, denoted $v \sim w$, if they are connected by an edge, i.e. $(v, w) \in E$. A *loop* is an edge that connects a vertex to itself. A graph is called a *simple graph* if it has no multiple edges and no loops.

Example 1.4 In Example 1.2, the vertices 1 and 2 are adjacent, there is a loop on the vertex 6 and a multiple edge between 4 and 5.

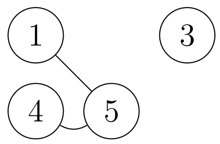
Definition 1.5. Let $G = (V, E)$ be a graph. A *subgraph* of G is a graph $G' = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$.

Example 1.6 Let $G = (V, E)$ be a graph.

(a) For $v \in V$, $G \smallsetminus v$ is the subgraph of G where you remove the vertex v and all edges containing v.

(b) For $e \in E$, $G \smallsetminus e$ is the subgraph of G where you remove the edge e. Notice that we do not remove any vertex.

(c) The following graph is a subgraph of the graph of Example 1.2.



Definition 1.7. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. A *morphism* of graphs $f : G_1 \to G_2$ is a pair of map (f_V, f_E) with $f_V : V_1 \to V_2$ and $f_E : E_1 \to E_2$ such that, for every $e = (v, w) \in E_1$, $f_E(e) = (f_V(v), f_V(w))$. A morphism $f = (f_V, f_E) : G_1 \to G_2$ is an *isomorphism* if there exists a morphism $g = (g_E, g_V) : G_2 \to G_1$ such that $f_V \circ g_V = g_V \circ f_V = \mathrm{id}$ and $f_E \circ g_E = g_E \circ f_E = \mathrm{id}$. We say then that G_1 and G_2 are *isomorphic*.

The relation "*is isomorphic to*" defines an equivalence relation on graphs. We usually study graphs up to isomorphisms by identifying a graph with its conjugacy class.

1.2 Directed graphs

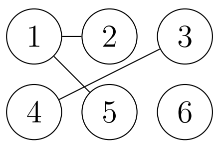
We can also considered edges with an orientation.

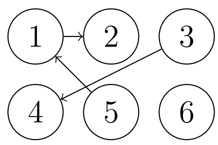
Definition 1.8. A *directed graph* is a graph $G = (V, E)$, together with applications

$$\partial_0 : E \longrightarrow V \text{ , and } \partial_1 : E \longrightarrow V$$

such that for every $e \in E$, $e = (\partial_0(e), \partial_1(e))$. An edge e of E is then called an *arrow* with *head* $\partial_1(e) \in V$ and *tail* $\partial_0(e) \in V$.

Example 1.9 The graph on the left is undirected, whereas the graph on the right is directed.





A directed subgraph of a directed graph G is then a subgraph of G with the induced orientation.

Definition 1.10. Let $G = (V, E)$ be a directed graph with orientation (∂_0, ∂_1). A *directed subgraph* of G is a subgraph, $G' = (V', E')$, of G with orientation $(\partial_0', \partial_1') = (\partial_0|_{E'}, \partial_1|_{E'})$.

In the same way, a morphism of directed graphs will be a morphism of graphs which preserve the orientation.

Definition 1.11. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two directed graphs. An *morphism of directed graphs* is an morphism of graph $f = (f_V, f_E) : G_1 \to G_2$ such that for every $e \in E_1$, $\partial_0(f_E(e)) = f_V(\partial_0(e))$ and $\partial_1(f_E(e)) = f_V(\partial_1(e))$. f is then an isomorphism if there is a morphism of directed graph $g = (g_V, g_E) : G_2 \to G_1$ such that $f_V \circ g_V = g_V \circ f_V = \mathrm{id}$ and $f_E \circ g_E = g_E \circ f_E = \mathrm{id}$.

1.3 Degrees and neighborhood

Definition 1.12. Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted by $\delta_G(v)$, is the number of edges that end at v:

$$\delta_G(v) = Card\{e \in E \mid v \text{ is at an end of } e\}.$$

We write $\delta(v)$ when the graph is understood. The *degree* of G is

$$\Delta(G) = \max_{v \in V} \delta_G(v).$$

Example 1.13 In the graph G of Example 1.2, $\delta(5) = 3$.

Definition 1.14. Let $G = (V, E)$ be a directed graph and let $v \in V$.

(a) The *positive degree*, denoted by $\delta_{G,+}(v)$, is the number of arrows pointing at v:

$$\delta_{G,+}(v) = \text{Card}\{e \in E \mid \partial_1(e) = v\}.$$

(b) The *negative degree*, denoted by $\delta_{G,-}(v)$, is the number of arrows pointing out from v:

$$\delta_{G,-}(v) = \text{Card}\{e \in E \mid \partial_0(e) = v\}.$$

We write $\delta_+(v)$ and $\delta_-(v)$ when the graph is understood.

Example 1.15 In the directed graph of Example 1.9, we have $\delta_+(1) = 1$ and $\delta_-(1) = 1$.

Proposition 1.16. *Let $G = (V, E)$ be a simple graph. Then*

$$\sum_{v \in V} \delta(v) = 2Card(E).$$

If G is directed, then

$$\sum_{v \in V} \delta_+(v) = \sum_{v \in V} \delta_-(v) = Card(E).$$

Proof. For the first one,

$$\begin{aligned}\sum_{v \in V} \delta(v) &= \sum_{v \in V} \text{Card}\{e \in E \mid v \text{ is at an end of } e\} \\ &= \sum_{v \in V} \sum_{u \in V} \delta_{(u,v) \in E}\end{aligned}$$

where $\delta_{(u,v) \in E}$ equals 1 if $(u, v) \in E$ and 0 else. Thus each edge (u, v) is counted twice, once for u and once for v. In particular,

$$\sum_{v \in V} \delta(v) = 2\text{Card}(E).$$

The case of a directed graph is the same but this time, as we consider the orientation on the edges, each edge is counted once in each sum. □

Definition 1.17. Let $G = (V, E)$ be a graph and $v \in V$. The *neighborhood*, $N_G(v)$, of v in G is the set of the vertices adjacent to v:

$$N_G(v) = \{w \in V \mid (v, w) \in E\}.$$

Example 1.18 In the graph G of Example 1.2, $N_G(5) = \{1, 4\}$.

Notice that in a graph $G = (V, E)$, if G is simple, for every vertex $v \in V$, we have $\text{Card}(N_G(v)) = \delta(v)$.

We can also define neighborhoods of vertices in a directed graph. We leave the definition as an exercise for the reader (because of the orientation, there are two neighborhoods to define $N_{G,+}(v)$ and $N_{G,-}(v)$).

1.4 Matrix representation

One useful way to represent a graph is the matrix representation.

Definition 1.19. Let $G = (V, E)$ be a graph and set $V = \{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* of G is the $n \times n$-matrix $A(G)$ with entries $a_{i,j}$ equal to the number of edges between v_i and v_j.

Notice that we have made a choice in the indexation of the vertices. Hence the adjacency matrix is in fact defined up to a permutation of the rows and columns (using the same permutation on the rows and the columns).

Here again, there is a definition of adjacency matrix for directed graph and the definition is left to the reader.

Example 1.20 For G the graph of Example 1.2,

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The whole information of the graph G can be read on the adjacency matrix.

Proposition 1.21. *Let $G = (V, E)$ be a graph, set $V = \{v_1, v_2, \ldots, v_n\}$ and let $A(G)$ be the adjacency matrix with entries $a_{i,j}$. Then, the following are satisfied.*

(i) $A(G)$ is symmetric.

(ii) G has no multiple edge if and only if for $1 \le i, j \le n$, $a_{i,j}$ equals 0 or 1.

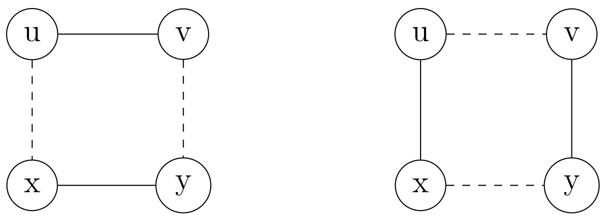
(iii) G has no loop if and only if for all $1 \le i \le n$, $a_{i,i} = 0$.

(iv) for $1 \le i \le n$, $\delta(v_i) = \sum_{j=1}^{n} a_{i,j}$.

(v) for $1 \le i, j \le n$, $v_i \in N_G(v_j)$ if and only if $a_{i,j} \neq 0$.

1.5 Switches

Definition 1.22. Let $G = (V, E)$ be a simple graph. Let $x, y, u, v \in V$ such that $(u, v), (x, y) \in E$ but $(u, x), (v, y) \notin E$. A *2-switch* with respect to the edges (u, v) and (x, y) replaces the edge (u, v) and (x, y) by (u, x) and (v, y).



If H is another simple graph, we say that G and H are *switch-equivalent* if H is obtained from G by a finite sequence of 2-switches. It defines an equivalence relation on graphs with the same set of vertices.

Notice that if two graphs G and H on the same set of vertices V are switch-equivalent, then for every $v \in V$, $\delta_G(v) = \delta_H(v)$. We will show that the reverse is also true.

Theorem 1.23 (Berge, 1973)**.** *Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two simple graphs with the same set of vertices. Then, G_1 and G_2 are switch-equivalent if and only if for all $v \in V$, $\delta_{G_1}(v) = \delta_{G_2}(v)$.*

For the proof of Theorem 1.23 we first need the following lemma.

Lemma 1.24. *Let $G = (V, E)$ be a graph. Let $E = \{v_1, v_2, \ldots, v_n\}$ with $\delta(v_1) \geq \delta(v_2) \geq \cdots \geq \delta(v_n)$. Then there is a graph G' switch-equivalent to G with $N_{G'}(v_1) = \{v_2, \ldots, v_{\delta(v_1)+1}\}$*

Proof. Set $d = \delta(v_1)$. Let i_0 be the minimal integer $1 \leq i \leq n$ such that $v_i \notin N_G(v)$.

If $i \geq d + 1$ then $G' = G$ is solution of Lemma 1.24.

Now fix $1 \leq \iota \leq d + 1$. Assume that Lemma 1.24 is true for $i_0 \geq \iota$ and assume $i_0 = \iota - 1$. Then, there is v_j adjacent to v_1 with $j \geq d + 2$. As $i_0 \leq j$, $\delta(v_{i_0}) \geq \delta(v_j)$. In particular, as $(v_1, v_j) \in E$, there is t such that $(v_{i_0}, v_t) \in E$ but $(v_j, v_t) \notin E$. Hence, we have four vertices v_1, v_j, v_{i_0}, v_t with $(v_1, v_j) \in E$, $(v_{i_0}, v_t) \in E$, $(v_1, v_{i_0}) \notin E$ and $(v_j, v_t) \notin E$. If we set H to be the graph obtain from G by a 2-switch with respect to (v_1, v_j) and (v_{i_0}, v_t) we have that $v_i \in N_H(v_1)$ for $1 \leq i \leq \iota$. By hypothesis, H, and thus G, is switch-equivalent to a graph H' with $N_{H'}(v_1) = \{v_2, \ldots, v_{\delta(v_1)+1}\}$.

Lemma 1.24 follows by induction. □

Proof of Theorem 1.23. If G_1 and G_2 are switch-equivalent, for every $v \in V$, $\delta_{G_1}(v) = \delta_{G_2}(v)$ (a switch does not change the degree of the vertices).

Conversely, assume that for every $v \in V$, $\delta_{G_1}(v) = \delta_{G_2}(v) = \delta(v)$ and let us prove that G_1 and G_2 are switch-equivalent by induction on $\Delta = \Delta(G_1) = \Delta(G_2)$. If $\Delta = 0$ there is nothing to prove. Assume that, for $\Delta \leq \Delta_0$, G_1 and G_2 are switch-equivalent. Assume then that $\Delta = \Delta_0 + 1$. Write $V = \{v_1, v_2, \ldots, v_n\}$ with $\Delta = \delta(v_1) \geq \delta(v_2) \geq \cdots \geq \delta(v_n)$. By Lemma 1.24, there is G_1' and G_2', switch-equivalent to respectively G_1 and G_2, such that $N_{G_1'}(v_1) = N_{G_2'}(v_1) = \{v_2, v_3, \ldots, v_{\delta(v_1)+1}\}$. Consider then $\overline{G_1} = G_1' \setminus v_1$ and $\overline{G_2} = G_2'$.

We have $\Delta(\overline{G_1}) = \Delta(\overline{G_2}) \leq \Delta - 1 = \Delta_0$ thus, by induction, $\overline{G_1}$ and $\overline{G_2}$ are switch-equivalent. Notice that, for $i = 1, 2$, any switch on $\overline{G_i}$ induces a switch on G'_i. In particular, G'_1 and G'_2 are switch-equivalent. Finally by transitivity, G_1 and G_2 are switch-equivalent. □

1.6 Exercises

Exercise 1.25. At a meeting, each person says hello to some other persons, but we don't know how many, and it depends on each person. Is the following sentence right?

"The number of persons who have said hello to an odd number of persons is an even number."

Exercise 1.26. Prove that, for every simple graph with $n \geq 2$ vertices, there are at least two vertices with the same degree.

Exercise 1.27. Can you connect 15 computers together in a way that every computer is connected to exactly 3 other computers?

Exercise 1.28. Can you find a simple graph with 7 vertices and sequence of degree $(7, 6, 5, 4, 3, 3, 2)$? what about the sequence of degree $(6, 6, 5, 4, 3, 3, 1)$?

Exercise 1.29. let $d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$ be natural numbers. Find a sufficient and necessary condition on the sequence $(d_1, d_2, \ldots, d_n)$ to ensure that there exists a graph (not necessarily simple) with n vertices and this sequence of degrees.

Exercise 1.30. If G is a non directed graph. How many possible directed graph can you get from G by choosing an orientation on G?

Exercise 1.31. Let G and H be two graphs and let $A(G)$ and $A(H)$ be their adjacency matrices. When can you say that G and H are isomorphic just looking at their adjacency matrices?

Chapter 2

Connectivity

2.1 Definitions

Definition 2.1. Let $G = (V, E)$ be a simple graph.

(a) A *walk* in G is a sequence of vertices $\gamma = (v_1, v_2, \ldots, v_n)$ such that, $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq n-1$. v_1 is called the *beginning* of γ and v_n is called the *end* of γ. We also say that γ is a walk from v_1 to v_n.

(b) If $\gamma = (v_1, \ldots, v_n)$ is a walk, the *length* of γ is $|\gamma| = n - 1$, i.e. the number of edges in γ.

(c) If $\gamma = (v_1, \ldots, v_n)$ is a walk, we denote by γ^{-1} the walk in the other direction: $\gamma^{-1} = (v_n, \ldots, v_1)$.

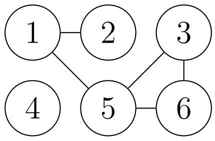
(d) G is *connected* if for every $v, w \in V$ there is a walk from v to w. When G is not connected, we say that G is *disconnected*.

(e) A *connected component* of G is a maximal connected subgraph of G.

(f) The *distance* $d_G(u, v)$ between two vertices $u, v \in E$ is the length of the shortest walk between those vertices. If there is no walk between the vertices, we let $d_G(u, v) = \infty$.

$$d_G(u, v) = \min\{|\gamma| \mid \gamma \text{ a walk from } u \text{ to } v\}.$$

Example 2.2 In the following graph, $(5, 1, 2)$ is a walk of length 2 from 5 to 2 and $(5, 6, 3, 5, 1)$ is a walk of length 4 from 5 to 1. There are 2 connected components.



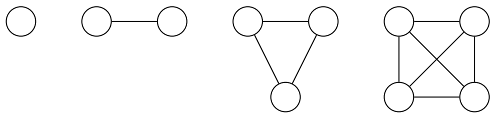
Lemma 2.3. A graph $G = (V, E)$ is connected if $d_G(u, v) < \infty$ for all $u, v \in V$.

Proof. By definition. □

Definition 2.4. Let $G = (V, E)$ be a simple graph. G is a *complete graph* if for every $v, w \in V$, $(v, w) \in E$.

For every $n > 0$, there is (up to isomorphism of graph) a unique complete graph with n vertices and we denote it by K_n.

Example 2.5 Here are K_1, K_2, K_3 and K_4.



Definition 2.6. Let $G = (V, E)$ be a simple graph. A *closed walk* is a walk which starts and ends at the same vertex. A *tree* is a connected graph without closed walk.

Proposition 2.7. *Let $G = (V, E)$ be a simple graph and $V = \{v_1, v_2, \dots, v_n\}$. If we denote by $A(G)^k$ the kth power of $A(G)$, and $a_{i,j}^k$ its coefficients, for $k \geq 1$, $a_{i,j}^k$ is equal to the number of walks of length k from v_i to v_j.*

Proof. See Exercise 2.22. □

Example 2.8 For G the graph of the Example 2.2,

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \qquad A(G)^2 = \begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 3 & 1 \\ 1 & 0 & 1 & 0 & 1 & 2 \end{bmatrix}$$

and

$$A(G)^3 = \begin{bmatrix} 0 & 2 & 1 & 0 & 4 & 1 \\ 2 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 2 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 4 & 0 & 2 & 4 \\ 1 & 1 & 3 & 0 & 4 & 2 \end{bmatrix}.$$

2.2 Bipartite graphs

Definition 2.9. A graph $G = (V, E)$ is called *bipartite* if V has a partition in two subsets V_1 and V_2 such that every edge $e \in E$ connects a vertex of V_1 and a vertex of V_2.

Theorem 2.10. *A graph $G = (V, E)$ is bipartite if and only if G has no odd closed walks (as subgraph).*

Proof. $\Rightarrow$ If G is (V_1, V_2)-bipartite, then so are all its subgraph. However, an odd closed walk C_{2k+1} is not bipartite.

$\Leftarrow$ Suppose that all closed walks in G are even. First, we note that it suffices to show the claim for connected graphs. Indeed, if G is disconnected, then each closed walk of G is contained in one of the connected components G_i. Assume that G is connected. Let $v \in V$ a chosen vertex, and define

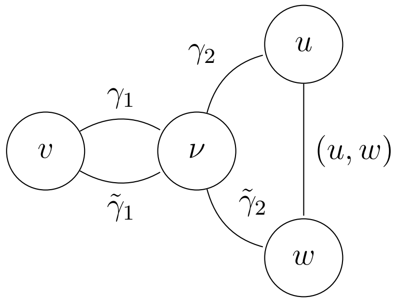
$$V_1 = \{x \in V \mid d_G(v, x) \text{ is even}\}$$

and

$$V_2 = \{y \in V \mid d_G(v, y) \text{ is odd}\} .$$

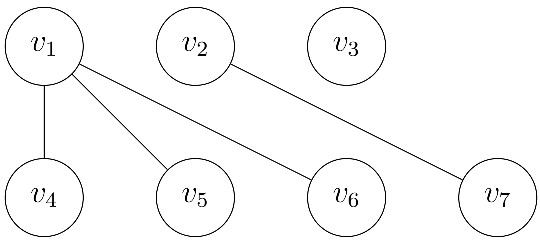
Since G is connected, $V = V_1 \cup V_2$. By definition of the distance, $V_1 \cap V_2 = \emptyset$.

Let $u, w \in V$ be both in V_1. Let γ and $\tilde{\gamma}$ be among the shortest paths from v to u and w. Assume that ν is the last common vertex of γ and $\tilde{\gamma}$. $|\gamma_1| = |\tilde{\gamma}_1|$ since γ and $\tilde{\gamma}$ are shortest paths,. Thus $|\gamma_2|$ and $|\tilde{\gamma}_2|$ have the same parity. $|\gamma_2| + |\tilde{\gamma}_2|$ is even, then $|(\gamma_2^{-1}, \tilde{\gamma}_2)|$ is even, so $(u, w) \notin E$ by assumption. Therefore V_1 and V_2 are stable subsets, and G is bipartite as claimed.



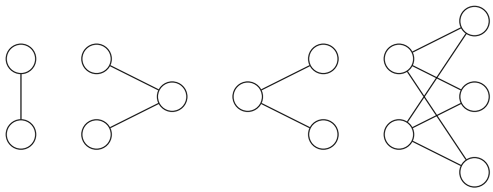
□

Example 2.11 The following graph is a bipartite graph, with for example $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5, v_6, v_7\}$.



Definition 2.12. Let $G = (V, E)$ be a simple bipartite graph with $V = V_1 \cup V_2$ the vertex partition, We say that G is a *complete bipartite graph* if for all $v_1 \in V_1$ and $v_2 \in V_2$, we have $(v_1, v_2) \in E$. if there are n vertices in V_1 and m vertices in V_2, then G is unique up to isomorphism and we denote it as $K_{n,m}$.

Example 2.13 Here are $K_{1,1}, K_{1,2}, K_{2,1}, K_{2,3}$.



One can also use the adjacency matrix to see if a graph is bipartite. Recall that the *transpose* of a $n \times m$-matrix A with entries $a_{i,j}$ is the $m \times n$-matrix A^t with entries $a'_{i,j} = a_{j,i}$.

Proposition 2.14. *(a) A graph is bipartite if and only if there exists a label for the vertices such that the adjacency matrix has the following block structure:*

$$\begin{bmatrix} 0 & A \\ A^t & 0 \end{bmatrix}$$

with A a matrix of size $n \times m$.

(b) A graph is a complete bipartite graph if and only if there exists a label for the vertices such that the adjacency matrix has the following form:

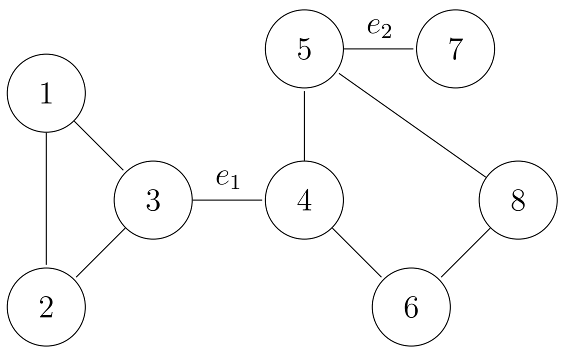
$$A(G) = \begin{bmatrix} 0 & \dots & 0 & 1 & \dots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & \dots & 1 \\ 1 & \dots & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & 1 & 0 & \dots & 0 \end{bmatrix}.$$

Proof. See Exercise 2.26. □

2.3 Bridges

Definition 2.15. An edge e is a *bridge* if and only if $G \setminus e$ is disconnected.

Example 2.16 In the next graph, e_1 and e_2 are bridges.



Theorem 2.17. *An edge $e \in G$ is a bridge if and only if e is not in any closed walk of G.*

Proof. $\Rightarrow$ If there is a closed walk in G containing e, say $C = (\gamma, e, \tilde{\gamma})$, then $(\tilde{\gamma}, \gamma)$ is a path in $G \setminus e$, and so e is not a bridge.

$\Leftarrow$ If $e = (u, v)$ is not a bridge, then u and v are in the same connected component of $G \setminus e$, and there is a walk γ from u to v in $G \setminus e$. Now, (e, γ) is a closed walk in G containing e. □

2.4 Exercises

Exercise 2.18. Let $G = (V, E)$ a simple graph. Prove that every vertex is contained in a unique connected component of G.

Exercise 2.19. Let $G = (V, E)$ a simple graph and let G_1 and G_2 be two connected components of G. Prove that $G_1 = G_2$ or $G_1 \cap G_2 = \emptyset$.

Exercise 2.20. Prove that if $G = (V, E)$ is a connected graph with n vertices, then G has at least $n - 1$ edges.

Exercise 2.21. Let $G = (V, E)$ be a graph. Prove that if G is simple and $\mathrm{Card(E)} > \binom{\mathrm{Card(V)}-1}{2}$ then G is connected. Is it still true if $\mathrm{Card(E)} = \binom{\mathrm{Card(V)}-1}{2}$?

Exercise 2.22. Prove Proposition 2.7.

Exercise 2.23. Let $G = (V, E)$ be a graph. Show that, if every vertex has degree at least 2, then G contains a closed walk.

Exercise 2.24. Let G a graph with n vertices. Show that the following properties are equivalent.

(i) G is a tree

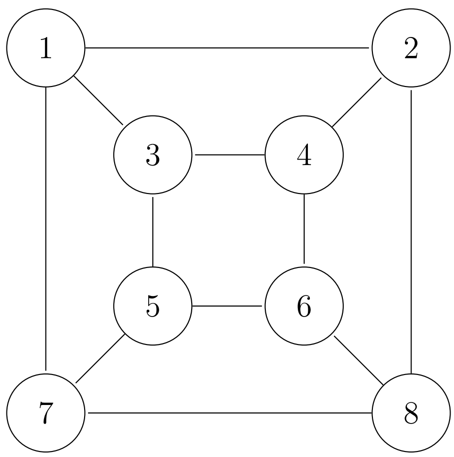
(ii) G is connected and has $n - 1$ edges

(iii) G has $n - 1$ edges and no closed walks

(iv) there exists only one walk between two vertices

(v) every edge of G is a bridge in G.

Exercise 2.25. Is the next graph bipartite?



Exercise 2.26. Prove Proposition 2.14.

Chapter 3

Planar graph

3.1 Definitions

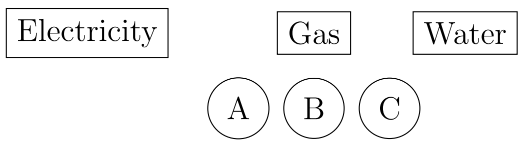
Definition 3.1. A *planar graph* $G = (V, E)$ is a graph which can be represented in the plane, i.e. such that we have a bijection $\phi : V \to \tilde{V} \subset \mathbb{R}^2$ which gives us a modified graph $\tilde{G} = (\tilde{V}, \tilde{E})$, where

$$\tilde{E} = \{(\phi(x), \phi(y)) \mid (x, y) \in E\}$$

in which the edges don't cross between them, in other words:

$$\forall (a, b) \neq (c, d) \in \tilde{E}, [a, b] \cap [c, d] = \emptyset \text{ or a point.}$$

Example 3.2 Consider three houses A, B, C and three factories, producing electricity, gas and water. Each house should join each factory. Is it possible to make those connections without any intersection?



Example 3.3

- Linking components on a printed circuit board.
- The following graph is planar, even if it isn't with this representation.

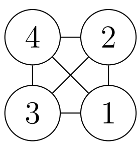


Figure 3.1: K_4

Definition 3.4. Let $G = (V, E)$ be a planar graph. A *face* in the graph G is a connected component of $\mathbb{R}^2 \setminus \{[x, y] \mid (x, y) \in E\}$.
The *dual graph* of G is the graph $G^* = (V^*, E^*)$ where V^* is the set of faces in G and there is an edge between the vertices $f_1 \in V^*$ and $f_2 \in V^*$ for every edge in G which separate the corresponding face of f_1 and the corresponding face of f_2.

Remark 3.5 If G is a finite graph represented in the plane, then there exists a unique unbounded face.

3.2 Euler's formula

One tool to say if a graph is not planar is the Euler Formula.

Theorem 3.6 (Euler, 1758)**.** *Let $G = (V, E)$ be a connected planar graph (in its planar representation), such that $V \neq \emptyset$. Let $v(G) = Card(V), e(G) = Card(E)$ and $f(G)$ be the number of faces of G. Then,*

$$v(G) - e(G) + f(G) = 2. \tag{3.1}$$

To prove this Theorem, we need the following lemma.

Lemma 3.7. Let $T = (V, E)$ be a tree. Then, for every $v \in V$, the connected components of $T \setminus v$ are trees.

Proof. Let T' be a connected component of $T \setminus v$. If T' contains a closed walk γ then γ is also a closed walk in T. This contradict the fact that T is a tree. Thus T' does not contains any closed walk and T' is a tree. □

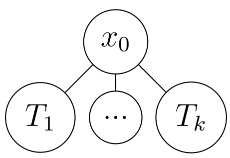
We can now prove Theorem 3.6.

Proof of Theorem 3.6. First we will prove it for a tree which has always a unique face (because it has no closed walks), then generalize it to any connected planar graph (non empty).

Assume then first that $G = T$ is a tree and fix $x \in V$. We will show by induction on Card(V) that (3.1) is satisfy.

Basic case: if T as one vertex, thus it has no edge. In particular $v(T) = 1$, $e(T) = 0$ and $f(T) = 1$, so the formula is correct.

Induction: Assume that, for $n \geq 1$, if Card$(V) \leq n$ then (3.1) is satisfied. Suppose now that $Card(V) = n + 1$ and consider $T \setminus v$. By Lemma 3.7 k sub-trees, $T \setminus v$ is a disjoint union of trees $(T_i)_{i=1,\dots,k}$ with a number of vertex less or equal to n.



We have also the following

$$\begin{aligned} v(T) - e(T) + f(T) &= 1 + \sum_{i=1}^{k} v(T_i) - (k + \sum_{i=1}^{k} e(T_i)) + 1 \\ &= 1 + \sum_{i=1}^{k} (v(T_i) - e(T_i) - 1) + 1 = 2. \end{aligned}$$

Thus, by induction, the property is satisfied for every tree.

Assume now that G is a graph and not necessarily a tree. We will do an induction on the number of faces of the graph, $f = f(G)$

Basic case: if $f(G) = 1$, then G is a tree, so the result was proved.

Induction: if $f(G) > 1$, then there exists an unbounded face which is separated of the rest of the plane by a closed walk $x_1 \sim x_2 \sim \ldots \sim x_n \sim x_1$. In the graph $\widehat{G} = (V, E \setminus \{(x_1, x_2)\})$ we have exactly the same faces, except for one face which has merged with the unbounded one, so $f(\widehat{G}) = f(G) - 1$. Seeing that $e(\widehat{G}) = e(G) - 1$, the result is proved by induction. □

The following proposition give a easy test to see if a graph is not planar.

Proposition 3.8. *Let $G = (V, E)$ be a simple connected planar graph with more than 3 vertices. Set $v(G) = Card(V), e(G) = Card(E)$. Then*

$$e(G) \leq 3v(G) - 6$$

Proof. See Exercise 3.17. □

3.3 Subdivisions of K_5 and $K_{3,3}$ and planar graphs

Definition 3.9. Let $G = (V, E)$ be a graph. An edge $e = (u, v)$ is *subdivided*, when it is replaced by a vertex w and two edges (u, w) and (v, w). A *subdivision* of G is a graph H

obtained from G by a finite sequence of subdivisions.

Planar graph behaves nicely with subdivisions.

Lemma 3.10. A graph is planar if and only if their subdivisions are planar.

Proof. It is clear that a subdivision of an edge will not change the fact that the graph is planar or not. □

The following theorem, due to Kuratowski, gives a very simple characterization of planar graphs. We can easily show that K_5 and $K_{3,3}$ are not planar (see Exercises 3.15 and 3.16). Thus every subdivision of K_5 and $K_{3,3}$ is not planar. Hence, every graph containing a subdivision of K_5 or $K_{3,3}$ as subgraph is not planar. Kuratowski proved that the converse is also true. The proof is a bit long and delicate. Therefore we just give here the result.

Theorem 3.11 (Kuratowski, 1930)**.** *A graph is planar if and only if it doesn't contain a subdivision of $K_{3,3}$ or K_5 as a subgraph.*

3.4 Exercises

Exercise 3.12. A country contains 11 cities which are linked either with train or road. Prove that necessarily, there exists a bridge with a railway getting up another, or a bridge with a road getting up another.

Exercise 3.13. Let $n \in \mathbb{N}$. We consider the graph $G_n = (V_n, E_n)$ with $V_n = \{1, .., n\}$ and $(x, y) \in E_n \Leftrightarrow (x + y)$ is prime.
Determine for which values of n G_n is planar.

Exercise 3.14.
1. Let G be a simple connected planar graph. Show that there exists a vertex with degree lower or equal to 5.
2. Is this result stay true if the graph is not connected anymore?

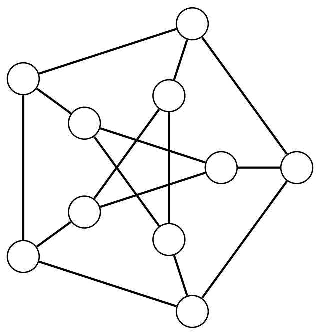
Exercise 3.15. Prove that K_5 is not planar.

Exercise 3.16. For which values of n and q is $K_{n,q}$ planar?

Exercise 3.17. Prove Proposition 3.8.

Exercise 3.18. What are all the graphs that can be represented on a sphere? What happens on the torus?

Exercise 3.19. Is the following graph, called the Petersen Graph, is planar?



Exercise 3.20. What is the analogue of Euler Formula for non connected graphs ?

Chapter 4

Eulerian paths and Hamiltonian paths

4.1 Eulerian graphs

Here we will work with simple graph to simplify the notations and the proofs but everything works the same for non simple graph. In particular, Euler Theorem (Theorem 4.3) is also true for non simple graph. The changes in the proof is left to the reader.

Definition 4.1. Let $G = (V, E)$ be a simple graph and $\gamma = (v_0, v_1, \ldots, v_n)$ be a walk in G. For $0 \leq i \leq n-1$ set $e_i = (v_i, v_{i+1}) \in E$.

(a) γ is a *trail* if for all $i \neq j$, $e_i \neq e_j$ (i.e. γ passes trough an edge at most once) and γ is a *closed trail* if moreover $v_1 = v_n$

(b) γ is an *Eulerian (closed) trail* if γ is a (closed) trail which passes trough every edge.

(c) G is *Eulerian* if there is an Eulerian closed trail in G.

Example 4.2

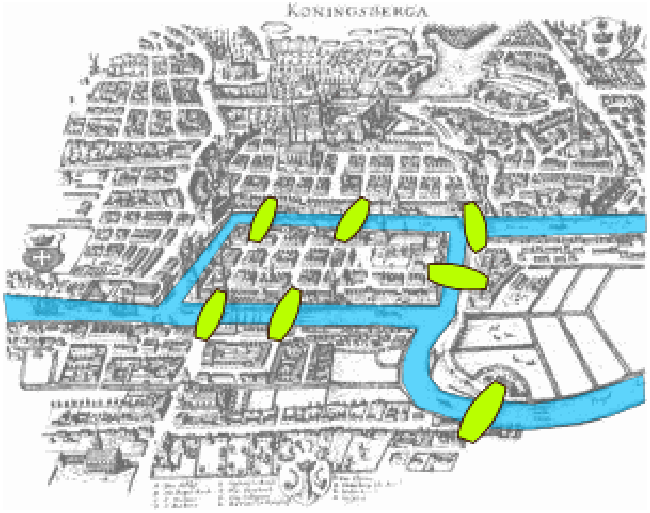
- K_3 is Eulerian but K_2 and K_4 are not Eulerian.
- If a simple graph G is Eulerian, then G is connected.

One famous problem of Eulerian graph is the Königsberg bridges problem.

To decide if a given graph is Eulerian or not may seem to be difficult at first sight. In fact there is an easy characterization which makes the problem almost trivial. The following theorem, called Euler's Theorem, was formulated by Euler in 1736 and a proof was first given by Hierholzer in 1873.

Theorem 4.3 (Euler Theorem, Hierholzer 1873)**.** *Let $G = (V, E)$ be a simple connected graph. Then G is Eulerian if and only if for all vertices $v \in V$, $\delta(v)$ is even.*

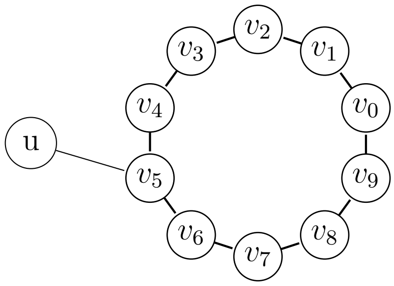
Proof. Assume first that G is Eulerian and let γ be an Eulerian closed trail on G which starts and ends at $u \in V$. Let $v \in V$ be a vertex with $v \neq u$ and let k be the number of times that v occurs in γ. As γ is an Eulerian closed trail, when you follow γ, every time an edge arrives at v there is another one which starts from v. In particular $\delta(v) = 2k$. Also, $\delta(u)$ is even, since γ starts and ends at u.



Königsberg bridges problem [1]

Assume now that every vertex has an even degree. Consider a trail $\gamma = (v_0, v_1, \ldots, v_n)$ in G which is maximal.

Then, for every $w \in N_G(v_n)$, γ already passes trough (v_n, w) (if not, γ may be extended in a longer trail). In particular, $v_n = v_0$. Indeed, if not and if v_n occurs k times in γ, then $\delta(v_n) = 2(k-1)+1$ but $\delta(v_n)$ is even. Hence, γ is a closed trail. Assume γ is not an Eulerian trail. Since G is connected, there is a vertex u such that $(v_i, u) \in E$ for some $0 \leq i \leq n$. Then, we can construct a trail $(u, v_i, v_{i+1}, \ldots, v_n, v_2, v_3, \ldots, v_{i-1})$ which is longer than γ.



This is a contradiction to the choice of γ ends the proof of Theorem 4.3. $\square$

With Theorem 4.3 it is now easy to decide if a given graph G is Eulerian or not. The next problem is then to construct an Eulerian closed trail in G. Indeed the previous proof is not constructive, i.e. it did not give an Eulerian closed trail in G. If the graph is small it may be possible to do it by hand but when the graph is enormous, it may be almost impossible. We then need an algorithm which can be done automatically, for example using a computer.

Algorithm 4.4 (Fleury)**.** Let $G = (V, E)$ be an Eulerian graph.

(E0) Choose a vertex $v_0 \in V$.

(E1) Repeat the following procedure for $i = 1, 2, \ldots$ as long as possible: suppose a trail $\gamma_i = (v_0, v_1, \ldots, v_{i-1})$ has been constructed. Choose $v_i \in N_G(v_{i-1})$ such that (v_{i-1}, v_i) is not already in γ_i and such that (v_{i-1}, v_i) is not a bridge of $G_i = G \setminus \{(v_0, v_1), (v_1, v_2), \ldots, (v_{i-2}, v_{i-1})\}$, unless there is no alternative.

Theorem 4.5. *Let G be a simple connected graph with at least 3 vertices. If G is Eulerian, then Algorithm 4.4 constructs an Eulerian closed trail.*

Proof. See Exercise 4.13 □

4.2 Hamiltonian graphs

Definition 4.6. Let $G = (V, E)$ be a simple graph and $\gamma = (v_0, v_1, \ldots, v_n)$ a walk in G.

(a) γ is an *Hamiltonian (closed) walk* if γ is a (closed) walk which passes through every vertex, $\{v_0, v_1, \ldots, v_n\} = V$.

(b) G is *Hamiltonian* if there is a Hamiltonian closed walk in G

Example 4.7

- For every $n \geq 3$, K_n is Hamiltonian but K_2 is not Hamiltonian.
- If a graph G is Hamiltonian, then G is connected.

Unlike for Eulerian graphs (Theorem 4.3), the problem is much more difficult, and there is no good characterization known for Hamiltonian graphs. It is actually still an open question to find one. Some characterization are known for small classes of graphs but not much more.

Notice that, given a simple graph G, if you add more edges to it to get a new simple graph G', then G' has "more chance" to be Hamiltonian than G. For example, if you add all the remaining possible edges, you get a complete graph which is then Hamiltonian. The following theorem gives a way to add edges to G which preserve the property of being Hamiltonian.

Theorem 4.8 (Ore, 1962)**.** *Let $G = (V, E)$ be a simple graph. Assume there exists $u, v \in V$ such that $(u, v) \notin E$ and*

$$\delta(u) + \delta(v) \geq Card(V).$$

Then, G is Hamiltonian if and only if $G \cup \{(u, v)\}$ is Hamiltonian.

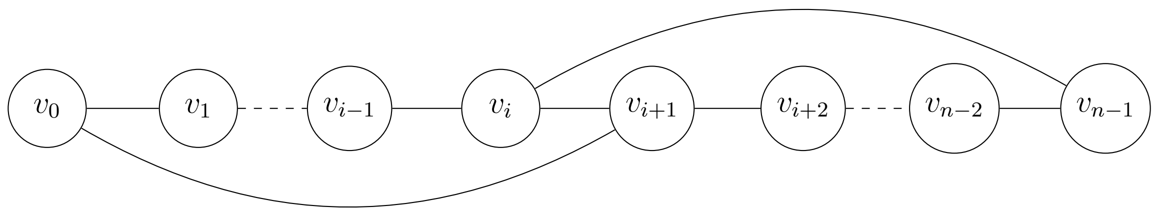
Proof. Let $n = \text{Card}(V)$ and let us fix $u, v \in V$ such that $\delta(u) + \delta(v) \geq n$.

Notice that a Hamiltonian closed walk in G is also a Hamiltonian closed walk in $G \cup \{(u, v)\}$. In particular, if G is Hamiltonian, then $G \cup \{(u, v)\}$ is Hamiltonian.

Assume now that $G \cup \{(u, v)\}$ is Hamiltonian. Let $\gamma = (v_0, v_1, \ldots, v_n)$ be a Hamiltonian closed walk such that $v_0 = v_n = u$. If γ does not pass through (u, v) then γ is a Hamiltonian closed walk of G. Thus we can assume that γ passes through (u, v) and we can assume that γ ends by (u, v), i.e. $v_{n-1} = v$. Then, $\gamma' = (u = v_0, v_1, \ldots, v_{n-1} = v)$ is a Hamiltonian path starting at u and ending at v. As $\delta(v) + \delta(u) \geq n$ there is $0 < i < n-1$ such that $(v, v_i) \in E$ and $(u, v_{i+1}) \in E$.

Then, $\overline{\gamma} = (u = v_0, v_1, \ldots, v_i, v = v_{n-1}, v_{n-2}, \ldots, v_{i+1}, u)$ is a Hamiltonian closed walk in G. □

Now using Theorem 4.8 iteratively we can get more and more edges till we get a complete graph or till there is no more couple of vertices $u, v \in V$ with $\delta(u) + \delta(v) \geq \text{Card}(V)$. For a graph $G = (V, E)$ and $u, v \in E$, we denote by $G \cup \{(u, v)\}$ the graph obtained by adding an edge between u and v, i.e. $G \cup \{(u, v)\} = (V, E \cup \{(u, v)\})$.



Algorithm 4.9. Let $G = (V, E)$ be a simple graph with $n = \text{Card}(V) \geq 3$. Define inductively a sequence $G_0, G_1, \ldots$ of simple graphs such that

(C1) $G_0 = G$, and

(C2) for $i \geq 1$, $G_{i+1} = G_i \cup \{(u, v)\}$ for $u, v \in V$ such that $\delta_{G_i}(u) + \delta_{G_i}(v) \geq n$.

This procedure stops when no new edge can be added.

In fact even if there may be more than one choice at some step of Algorithm 4.9, we always end up with the same graph.

Lemma 4.10. *Let G be a simple graph with $n = \text{Card}(V) \geq 3$. The graph constructed from G by Algorithm 4.9 is unique.*

Proof. Assume that Algorithm 4.9 can give two different graphs,

$$H = G \cup \{e_1, e_2, \ldots, e_k\} \text{ and } H' = G \cup \{f_1, f_2, \ldots, f_s\}$$

where the edges are added in the given order. Set $H_i = G \cup \{e_1, \ldots, e_i\}$ and $H'_i = G \cup \{f_1, \ldots, f_i\}$. For $i = 0$, $H_0 = H'_0 = G$, and let $e_k = (u, v)$ be the first edge such that $e_k \neq f_i$ for all i. Then, $\delta_{H_{k-1}}(u) + \delta_{H_{k-1}}(v) \geq n$ since $e_k \in H_k$ but $e_k \notin H_{k-1}$. By the choice of e_k, we have $H_{k-1} \subseteq H'$ and thus also $\delta_{H'}(u) + \delta_{H'}(v) \geq n$ which means that e_k must be in H', which is a contradiction. Therefore, $H \subseteq H'$. Symmetrically, we deduce that $H' \subseteq H$ and thus $H = H'$. □

Definition 4.11. Let $G = (V, E)$ be a simple graph with $\text{Card}(E) \geq 3$. The graph obtained from G by Algorithm 4.9 is called the *closure* of G and is denoted $\mathfrak{cl}(G)$.

Theorem 4.12. *Let $G = (V, E)$ be a simple graph with Card$(E) \geq 3$. G is Hamiltonian if and only if its closure is Hamiltonian.*

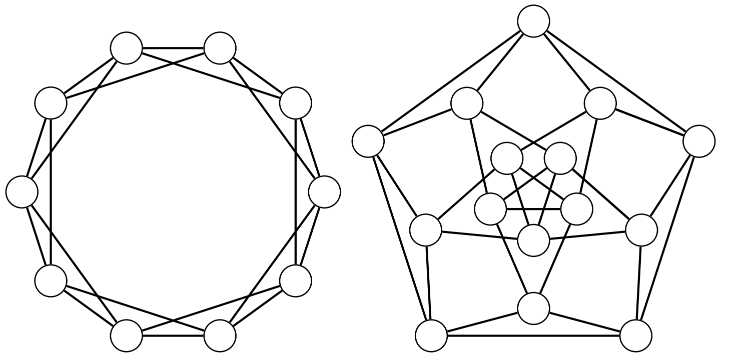
Proof. This is an easy corollary from Theorem 4.8 □

Thus, to see if a graph G is Hamiltonian, one way is to get its closure $\mathfrak{cl}(G)$, which can be done using a computer, and then try to see if $\mathfrak{cl}(G)$ is Hamiltonian. The problem is of course that we may end up with a graph which is not complete and where it is still difficult to see that the graph is Hamiltonian. Also, Algorithm 4.9 is useless if you do not have a couple of vertices $u, v \in V$ with $\delta(u) + \delta(v) \geq \text{Card}(V)$. Indeed, in that case $\mathfrak{cl}(G) = G$.

4.3 Exercises

Exercise 4.13. Prove Theorem 4.5: Fleury's Algorithm constructs an Eulerian closed trail.

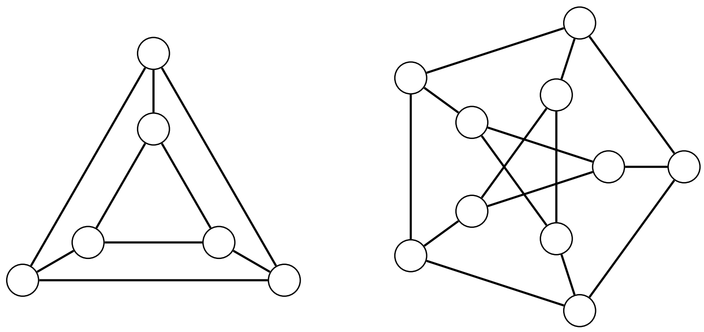
Exercise 4.14. Can you find an Eulerian trail in the following graphs?



Exercise 4.15. Find a necessary and sufficient condition for a connected graph to have an Eulerian trail.

Exercise 4.16. Find an analogue of Euler's Theorem for directed graphs.

Exercise 4.17. Are these graphs Hamiltonian? (The second graph again the Petersen Graph)



Exercise 4.18. Find m and n such that $K_{m,n}$ is Hamiltonian.

Exercise 4.19. Let $G = (V, E)$ be a graph and set $n = \text{Card(V)}$. Prove that, if $n \geq 3$ and $\min_{v \in V}(\delta(v)) \geq n/2$ then G is Hamiltonian.

Exercise 4.20. Let $G = (V, E)$ be a graph and $S \subseteq V$ a subset of vertices. Prove that, if G is Hamiltonian then $G \setminus S$ has less or equal than $\text{Card}(S)$ connected components.

Chapter 5

Graph coloring

5.1 Edge colorings

5.1.1 Definitions

Definition 5.1. Let $G = (V, E)$ a graph.

A *k-edge coloring* $\alpha : E \to [1, k]$ is an assignment of k colors to its edges. We write G^α to indicate that G has the edge coloring α.

The coloring α is *proper* if no two adjacent edges obtain the same color: $\alpha(e_1) \neq \alpha(e_2)$ for adjacent e_1 and e_2.

The *edge chromatic number* $\chi'(G)$ is defined as

$$\chi'(G) = \min\{k \mid \text{ there exists a proper } k\text{-edge coloring of } G\}.$$

A vertex $v \in V$ and a color $i \in [1, k]$ are *incident* with each other, if there exists $u \in V$ such that $(u, v) \in E$ and $\alpha((v, u)) = i$. If $v \in V$ is not incident with a color i, then i is *available* for v.

Proposition 5.2. *We have the following inequalities.*

$$\Delta(G) \leq \chi'(G) \leq Card(E).$$

Proof. See Exercise 5.22. □

Example 5.3 The three numbers in Proposition 5.2 could be equal. For example, this happens when G is a star. But often the inequalities are strict.

Remark 5.4 In a graph $G = (V, E)$ with a k-coloring α, the coloring can be thought as a partition $\{E_1, \ldots, E_k\}$ of E where $E_i = \{e \in E \mid \alpha(e) = i\}$.

5.1.2 Optimal coloring

We will show that for bipartite graphs, the lower bound is always optimal: $\Delta(G) = \chi'(G)$.

Lemma 5.5. *Let G be a connected graph that is not an odd closed walk. Then there exists a 2-edge coloring (that not need to be proper), in which both colors are incident with each vertex v with $\delta_G(v) \geq 2$.*

Proof. Assume that G is non trivial.

- Suppose first that G is Eulerian. If G is an even closed walk, then a 2-edge coloring exists as required. Otherwise, since now $\delta_G(v)$ is even for all $v \in V$ (see Theorem 4.3), G has a vertex v_1 with $\delta_G(v_1) \geq 4$.

  Let $(e_1, \ldots e_t)$ be an Eulerian closed trail of G, where $e_i = (v_i, v_{i+1})$. Define

  $$\alpha(e_i) = \begin{cases} 1 & \text{if i is odd;} \\ 0 & \text{if i is even.} \end{cases}$$

  Hence the claim holds.

- Suppose then that G is not Eulerian. We define a new graph G_0 by adding a vertex v_0 to G and connecting v_0 to each $v \in G$ of odd degree (see Exercise 1.25). In G_0 every vertex has even degree, and hence G_0 is Eulerian. By the previous case, there is a required coloring α of G_0 above. Now, α restricted to E is a coloring of G as required by the claim, since each vertex v_i with odd degree $\delta_G(v_i) \geq 3$ is entered and departed at least one in the Eulerian closed trail by an edge of the original graph.

□

Definition 5.6. For a k-edge coloring α of $G = (V, E)$, let

$$c_\alpha(v) = Card\left(\{i \text{ such that } v \text{ is incident with } i \in [1, k]\}\right).$$

A k-edge coloring β is an *improvement* of α if

$$\sum_{v \in V} c_\beta(v) > \sum_{v \in V} c_\alpha(v).$$

Also, α is *optimal* if it cannot be improved.

Lemma 5.7. *An edge coloring α of $G = (V, E)$ is proper if and only if $c_\alpha(v) = \delta_G(v)$ for all vertices $v \in V$.*

Remark 5.8 A graph G always has an optimal k-edge coloring, but maybe not any proper k-edge coloring.

Lemma 5.9. *Let α be an optimal k-edge coloring of $G = (V, E)$, and let $v \in V$. Suppose that the color i is available for v, and the color j is incident with v at least twice. Then the connected component H of $G^\alpha[i, j] = (V, E_i \cup E_j)$ that contains v is an odd closed walk.*

Proof. Suppose that the connected component H is not an odd closed walk. By Lemma 5.5, H has a 2-edge coloring $\tilde{\alpha} : E \to \{i, j\}$ in which both i and j are incident with each vertex x with $\delta_H(x) \geq 2$.

We obtain a recoloring β of G as follows:

$$\begin{aligned} \beta(e) = &\tilde{\alpha}(e) \text{ if } e \in H \\ &\alpha(e) \text{ if } e \notin H. \end{aligned}$$

Since $\delta_H(v) \geq 2$, and in β both colors i and j are now incident with v, $c_\beta(v) = c_\alpha(v) + 1$. Furthermore, by the construction of β, we have $c_\beta(u) \geq c_\alpha(u)$ for all $u \neq v$. Therefore, $\sum_{u \in V} c_\beta(u) > \sum_{u \in V} c_\alpha(u)$, which contradicts the optimality of α. Hence H is an odd closed walk. □

Theorem 5.10 (König, 1916)**.** *If G is bipartite, then*

$$\Delta(G) = \chi'(G).$$

Proof. Let α be an optimal $\Delta(G)$-edge coloring of $G = (V, E)$. If there is a $v \in V$ with $c_\alpha(v) < \delta_G(v)$, then by Lemma 5.9, G would contain an odd closed walk. But a bipartite graph does not contain such closed walks. Therefore, for all vertices v, $c_\alpha(v) = \delta_G(v)$. By Lemma 5.7, α is a proper coloring, and $\Delta(G) = \chi'(G)$ as required. □

5.1.3 Vizing's theorem

Theorem 5.11 (Vizing, 1964)**.** *For any graph G,*

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

Proof. a bit long and complicate ... □

Remark 5.12 Vizing's theorem (nor its present proof) does not offer any characterization for the graphs for which

$$\chi'(G) = \Delta(G) + 1.$$

In fact, it is one of the famous open problems of graph theory to find such a characterization. The answer is known (only) for some special classes of graphs.

5.2 Vertex colorings

The vertices of a graph G can also be classified using colorings. These colorings tell that some vertices have a common property if they share the same color.

5.2.1 Definitions

Definition 5.13. A *k-vertex coloring* of a graph $G = (V, E)$ is a mapping $\alpha : V \to [1, k]$. The coloring α is *proper* if adjacent vertices obtain a different color: for all $(u, v) \in E$, we have $\alpha(u) \neq \alpha(v)$. A color $i \in [1, k]$ is said to be *available* for a vertex v if no neighbor of v is colored by i.

A graph G is *k-colorable* if there is a proper k-coloring for G. The *vertex chromatic number* $\chi(G)$ of G is defined as

$$\chi(G) = \min\{k \mid \text{ there exists a proper } k\text{-vertex coloring of } G\}$$

If $\chi(G) = k$, then G is k-chromatic.

Remark 5.14

- If a graph is colorable, then it obviously can not have loops.
- Parallel edges can be reduced to one.

So we may assume our graphs here to be simple.

5.2.2 Looking for the chromatic number

Remark 5.15 We could compute that

$$\chi(K_4) = 4$$

Theorem 5.16. *A graph G is 2-colorable if and only if it is bipartite.*

Proof. Each proper vertex coloring $\alpha : V_G \to [1, k]$ provides a partition $\{V_1, \ldots, V_k\}$ of the vertex set V, where $V_i = \{v \in V \mid \alpha(v) = i\}$. □

Theorem 5.17 (The four-color theorem)**.** *Every simple planar graph is 4-colorable.*

Proof. The only known proofs require extensive computer runs, then we do not develop it here. The first such proof was obtained by Kenneth Appel and Wolfgang Haken in 1976. □

Nevertheless, in Exercise 5.23 and Exercise 5.24 we prove the result for respectively 6-colorability and 5-colorability.

5.2.3 Brooks' theorem

Lemma 5.18. For all graph $G = (V, E)$,

$$\chi(G) \leq \Delta(G) + 1.$$

Proof. Let $V = \{v_1, \ldots, v_n\}$, and define $\alpha : V \to \mathbb{N}$ inductively as follows: $\alpha(v_1) = 1$, and

$$\alpha(v_i) = \min\{j \mid \alpha(v_t) \neq j \text{ for all } t < i \text{ with } (v_i, v_t) \in E\}.$$

Then, α is proper and

$$\alpha(v_i) \leq \delta_G(v_i) + 1$$

for all i. □

We will see that the maximum value $\Delta(G) + 1$ is obtained only in two special cases, as it was shown by Brooks in 1941.

Theorem 5.19. *Let G be a connected graph. Then $\chi(G) = \Delta(G) + 1$ if and only if either G is an odd closed walk or a complete graph.*

Example 5.20 Suppose we have n objects $V = \{v_1, \ldots, v_n\}$, some of which are not compatible (like chemicals that react with each other, or worse, graph theorists who will fight during a conference). In the storage problem we would like to find a partition of the set V with as few classes as possible such that no class contains two incompatible elements. In graph theoretical terminology we consider the graph $G = (V, E)$, where $(v_i, v_j) \in E$ just in case v_i and v_j are incompatible, and we would like to color the vertices of G properly using as few colors as possible. This problem requires that we find $\chi(G)$.

Remark 5.21 Unfortunately, no good algorithms are known for determining $\chi(G)$.

5.3 Exercises

Exercise 5.22. Prove the Proposition 5.2.

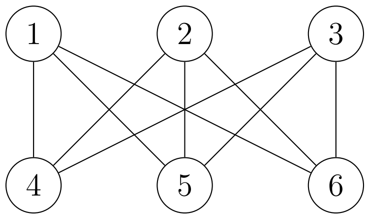
Exercise 5.23. Let G be planar graph with n vertices. Prove that $\chi(G) \leq 6$.

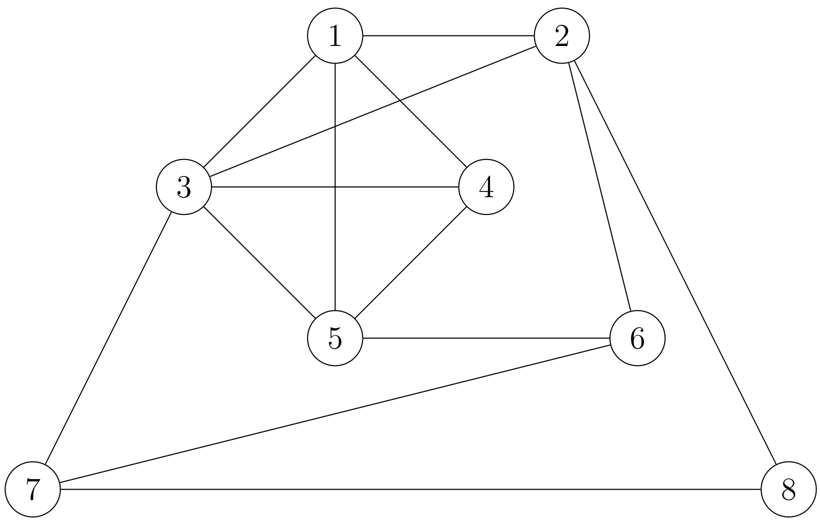
Exercise 5.24. Let G be planar graph with n vertices. Prove that $\chi(G) \leq 5$.

Exercise 5.25. Assume that between two persons, only two relationships exist: friend or enemy. We assume here that the relationships are symmetric. We consider a group with 18 persons. Prove that there exists a group of 4 persons which have the same relationship.

Exercise 5.26. Assume that between two persons, only three relationships exist: friend, enemy or unknown. We assume here that the relationships are symmetric. We consider a group with 17 persons. Prove that there exists a group of 3 persons which have the same relationship.

Exercise 5.27. Compute the edge chromatic numbers and the vertex chromatic numbers of the following graphs.





Exercise 5.28. We should organize several tests over a week, taking the minimum time. Each test corresponds to a half-day. There are 7 tests to plan, numerated from 1 to 7. The following courses could not be planned in the same time: and 1 and 2, 1 and 3, 1 and 4, 1 and 7, 2 and 3, 2 and 4, 2 and 5, 2 and 7, 3 and 4, 3 and 6, 3 and 7, 4 and 5, 4 and 6, 5 and 6, 5 and 7 and finally 6 and 7. During how many days the students will have tests?

Chapter 6

Automata

6.1 Deterministic finite automata

Definition 6.1. An *alphabet* Σ is a finite set and the elements of an alphabet are called *symbol.*

A *word* on Σ is a sequence $w = a_1 a_2 \cdots a_n$ of symbols $a_i \in \Sigma$ for $n \geq 0$, $1 \leq i \leq n$. For $n = 0$, $w = \emptyset$ is the empty word on Σ. We denote by $\Sigma^\star$ the set of all words on the alphabet Σ:

$$\Sigma^\star = \{a_1 a_2 \cdots a_n \mid n \geq 0,\ a_i \in \Sigma\}\,.$$

A *language* $\mathcal{L}$ on Σ is a subset of $\Sigma^\star$.

Example 6.2 $\Sigma = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$ is an alphabet and $\mathcal{L} = \{this, is, a, language\}$ is a language on Σ.

$\Sigma' = \{:, ;, 8, (,), p\}$ is an alphabet and $\mathcal{L}' = \{:), :(, 8), ;), :p\}$ is a language on Σ'.

Definition 6.3. A *deterministic finite automaton* is a 5-tuple $(Q, \Sigma, q_0, F, \tau)$ where

(i) Q is a finite set called the *set of states*,

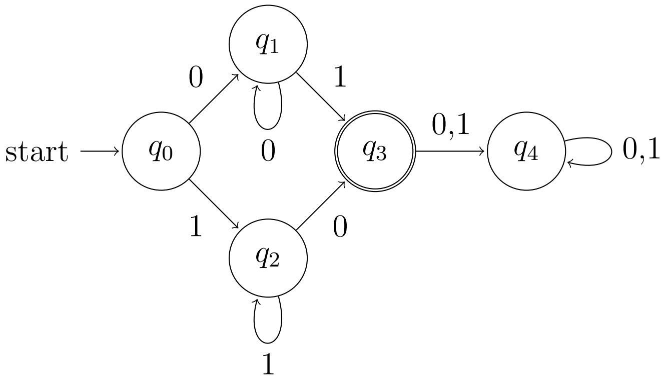
(ii) Σ is a finite set called *alphabet*,

(iii) $q_0 \in Q$ is the *start state*,

(iv) $F \subseteq Q$ is the set of *accepting states*,

(v) $\tau : Q \times \Sigma \longrightarrow Q$ is the *transition function.*

Example 6.4 Here is an example of a deterministic finite automaton with set of states $Q = \{q_0, q_1, q_2, q_3\}$, alphabet $\Sigma = \{0, 1\}$, start state q_0 and set of accepting states $F = \{q_3\}$. The transition function τ is representing by the arrows labeled by 1 or 0. For example, $\tau(q_0, 0) = q_1$ and $\tau(q_0, 1) = q_2$.



An automaton is a special case of a Turing machine. Turing machines were introduced by Alan Turing in 1936 and one of the most famous Turing machine is a computer. Automaton are used to recognized words and languages.

Algorithm 6.5. Let $A = (Q, \Sigma, q_0, F, \tau)$ be a deterministic finite automaton and $w \in \Sigma^\star$ a word.

(R0) Start at q_0.

(R1) For every input symbol a_i in the sequence w, use the transition function τ to move from state to state.

(R2) If after that all symbols in w are consumed, the current state is an accepting one (is in F), then A recognizes w. Else, w is rejected.

Example 6.6 The automaton of Example 6.2 recognizes 001 but not 010.

Definition 6.7. For $A = (Q, \Sigma, q_0, F, \tau)$ a deterministic finite automaton, we denote by $\mathcal{L}(A)$ the language recognized by A, i.e. $\mathcal{L}(A)$ is the set of all words $w \in \Sigma^\star$ which are recognized by A using Algorithm 6.5.

Example 6.8 For A the automaton of Example 6.2, $\mathcal{L}(A)$ is the language of words of the form $00\cdots01$ with at least one 0 or $11\cdots10$ with at least one 1.

6.2 Non-deterministic finite automata

Roughly speaking, a non-deterministic finite automaton is an automaton where we can be in more than one state, or even no state, at the same time.

Definition 6.9. A *non-deterministic finite automaton* is a a 5-tuple $(Q, \Sigma, q_0, F, \tau)$ where

(i) Q is a finite set called the *set of states*,

(ii) Σ is a finite set called *alphabet*,

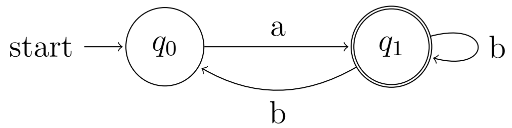
(iii) $q_0 \in Q$ is the *start state*,

(iv) $F \subseteq Q$ is the set of *accepting states*,

(v) $\tau : Q \times \Sigma \longrightarrow \{\text{subset of } Q\}$ is the *transition function*.

Notice that $\emptyset$ is a subset of Q. Thus, from one state and with a given letter, we can arrive in no state.

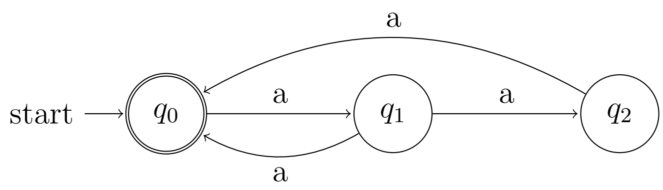
Example 6.10 (a) Here is a non-deterministic finite automaton.



There are two states q_0 and q_1, and the transition function δ is $\delta(q_0, a) = \{q_1\}$, $\delta(q_0, b) = \delta(q_1, a) = \emptyset$ and $\delta(q_1, b) = \{q_0, q_1\}$.

(b) A finite deterministic automaton is a finite non-deterministic automaton.

(c) Here is another non-deterministic finite automaton on the alphabet $\Sigma = \{a, b\}$.



Algorithm 6.11. Let $A = (Q, \Sigma, q_0, F, \tau)$ be a deterministic finite automaton and $w \in \Sigma^\star$ a word.

(R0) Start at q_0.

(R1) For every input symbol a_i in the sequence w, use the transition function τ to move from states to states (you can be in no state or more than one state at a time).

(R2) If after that all symbols in w are consumed, one of the current states is an accepting one (is in F), then A recognizes w. Else, w is rejected.

Example 6.12 For the automaton A of Example 6.10(c), $\mathcal{L}(A)$ is the language of all words with only the symbol a and the number of a is of the form $2k + 3l$ where $k, l \in \mathbb{N}$. Is it the same as all the words with only the letter a and different to the word a?

Theorem 6.13. *Let Σ be an alphabet and $\mathcal{L}$ be a language. The following propositions are equivalent.*

(i) There is a deterministic finite automaton A such that $\mathcal{L} = \mathcal{L}(A)$.

(ii) There is a non-deterministic finite automaton A such that $\mathcal{L} = \mathcal{L}(A)$.

Proof. $(i) \Rightarrow (ii)$ because a deterministic finite automaton is a non-deterministic finite automaton.

Conversely, assume there is a non-deterministic finite automaton $A = (Q, \Sigma, q_0, F, \tau)$ such that $\mathcal{L} = \mathcal{L}(A)$. The idea is to construct a deterministic automaton from A which has the same language. Let

$$Q' = \{\text{subset of } Q\},$$
$$q_0' = \{q_0\} \in Q',$$
$$F' = \{U \in Q' \mid U \cap F \neq \emptyset\},$$

and define $\tau' : Q' \times \Sigma \longrightarrow$ as the following. For a subset $U \in Q'$ and $a \in \Sigma$, set

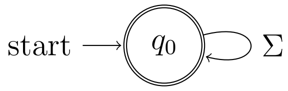
$$\tau'(U, a) = \bigcup_{q \in U} \tau(q, a) \ \in Q'.$$

Then $A' = (Q', \Sigma, q_0', F', \tau')$ is a deterministic finite automaton and it is easy to see that $\mathcal{L}(A') = \mathcal{L}(A) = \mathcal{L}$. □

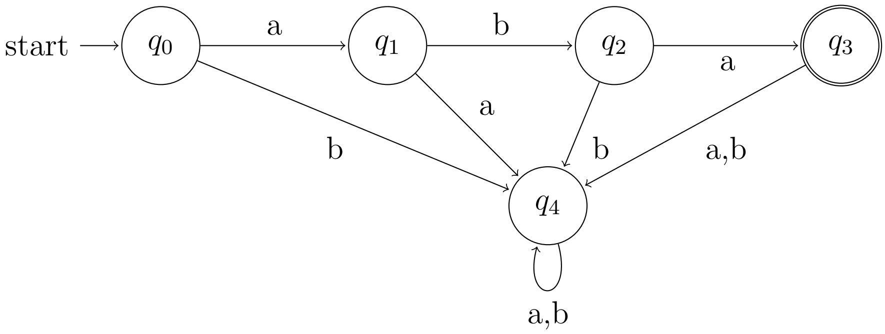
6.3 Regular language

Definition 6.14. A language $\mathcal{L}$ is *regular* if there is an automaton A such that $\mathcal{L} = \mathcal{L}(A)$.

Example 6.15
- The language $\mathcal{L} = \emptyset$ is regular because $\emptyset = \mathcal{L}(A)$ for A an automaton without accepting state.
- For Σ an alphabet, $\Sigma^\star$ is regular.



- The language $\mathcal{L} = \{aba\}$ on $\Sigma = \{a, b\}$ is regular.



- More generally any language with only one word is regular (Exercise 6.23).

Here are some basic constructions with languages.

Definition 6.16. Let Σ be an alphabet and $\mathcal{L}_1$ and $\mathcal{L}_2$ be two languages.

1. The *union* of $\mathcal{L}_1$ and $\mathcal{L}_2$ is the language
$$\mathcal{L}_1 \cup \mathcal{L}_2 = \{w \in \Sigma^\star \mid w \in \mathcal{L}_1 \text{ or } w \in \mathcal{L}_2\}.$$
2. The *intersection* of $\mathcal{L}_1$ and $\mathcal{L}_2$ is the language
$$\mathcal{L}_1 \cap \mathcal{L}_2 = \{w \in \Sigma^\star \mid w \in \mathcal{L}_1 \text{ and } w \in \mathcal{L}_2\}.$$
3. the *complementary* of $\mathcal{L}_1$ is the language,
$$\mathcal{L}_1^c = \{w \in \Sigma^\star \mid w \notin \mathcal{L}_1\}.$$
4. For a word $w = s_1 s_2 \cdots s_k \in \Sigma^\star$ we denote by w^{-1} the word $s_k s_{k-1} \cdots s_1$. Then, the *reversal* language of $\mathcal{L}_1$ is the language
$$\mathcal{L}_1^{-1} = \{w^{-1} \mid w \in \mathcal{L}_1\}.$$
5. The *concatenation* of $\mathcal{L}_1$ and $\mathcal{L}_2$ is the language
$$\mathcal{L}_1\mathcal{L}_2 = \{w_1 w_2 \mid w_1 \in \mathcal{L}_1 \text{ and } w_2 \in \mathcal{L}_2\}.$$

6. The *closure* of $\mathcal{L}_1$ is the language

$$\mathcal{L}^\star = \{w_1 w_2 \cdots w_n \mid n \in \mathbb{N}, w_i \in \mathcal{L}_1\}.$$

The following theorem state that the class of regular languages is stable by these constructions.

Theorem 6.17. *Let Σ be an alphabet and $\mathcal{L}_1$ and $\mathcal{L}_2$ be two languages. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are regular, then*

(i) $\mathcal{L}_1 \cup \mathcal{L}_2$ *is regular,*

(ii) $\mathcal{L}_1 \cap \mathcal{L}_2$ *is regular,*

(iii) $\mathcal{L}_1^c$ *is regular,*

(iv) $\mathcal{L}_1^{-1}$ *is regular,*

(v) $\mathcal{L}_1\mathcal{L}_2$ *is regular,*

(vi) $\mathcal{L}_1^\star$ *is regular.*

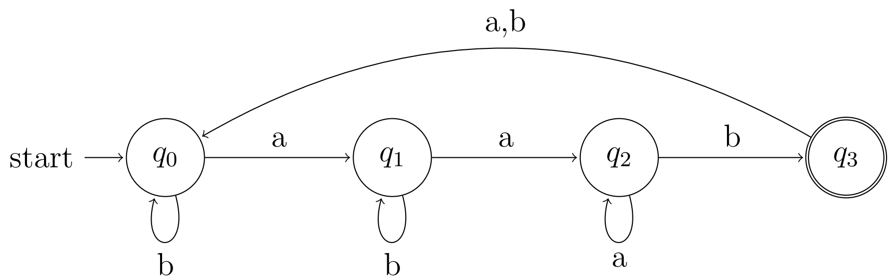
Proof. See Exercise 6.25. □

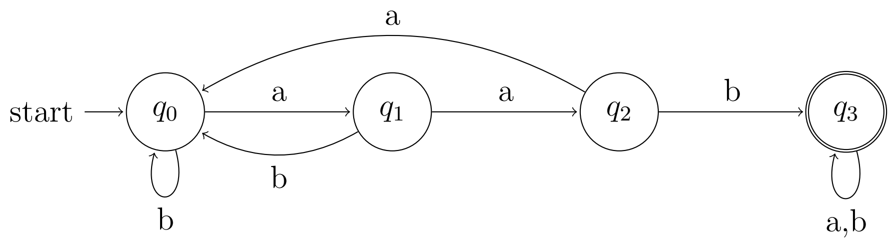
6.4 Exercises

Exercise 6.18. Let $\Sigma = \{a, b\}$. Find a (non-)deterministic finite automaton A such that $\mathcal{L}(A) = \{(ab)^n \mid n \in \mathbb{N}\}$. Here $(ab)^n = abab \dots ab$ with n-time the word ab.

Exercise 6.19. Let $\Sigma = \{a, b\}$. Find a (non-)deterministic finite automaton such that $\mathcal{L}(A)$ is the language of all the words of $\Sigma^\star$ which contains the word ab.

Exercise 6.20. Let $\Sigma = \{a, b\}$, find the language of the following automata.





Exercise 6.21. Find the language of the automata described in Example 6.10.

Exercise 6.22. Find finite deterministic automata with the same language as the non-deterministic automata of Example 6.10.

Exercise 6.23. Let Σ be an alphabet and $w \in \Sigma^\star$ a word. Show that the language $\mathcal{L} = \{w\}$ is regular.

Exercise 6.24. Let $\Sigma = \{a, b\}$ and

$$\mathcal{L} = \{w \in \Sigma^\star \mid w \text{ is not of the form } a^n b^n, \text{ for all } n \geq 0\}.$$

Show that $\mathcal{L}$ is not regular.

Exercise 6.25. Prove Theorem 6.17.

Chapter 7

Random walks on graphs

7.1 Generalities

In this section, for $z \in \mathbb{R}^d$, we set $|z|_1 := \sum_{k=1}^{d} |z_k|$.

Definition 7.1. Let $G = (V, E)$ be a simple graph. A *transition probability* is a map $a : E \to [0, 1]$ such that for all $x \in V$,

$$\sum_{y \sim x} a_{(x,y)} = 1.$$

A *random walk* on G with transition probability a starting at x is a random variable $(X_n)_{n \in \mathbb{N}}$ such that

$$\begin{cases} X_0 = x \text{ almost surely,} \\ \mathbb{P}(X_{n+1} = y | X_n = x) = a_{(x,y)}. \end{cases}$$

The following example can be seen as the canonical example.

Example 7.2 The simple random walk on $\mathbb{Z}^d$ is the random walk with $a_{(x,y)} = \frac{\delta_{x \sim y}}{\delta(x)}$ where $\delta_{x \sim y}$ equal 1 if $x \sim y$ and 0 else. In this case $\delta(x) = 2d$ and $x \sim y$ if and only if $|x - y|_1 = 1$ (i.e. $x \sim y$ if and only if there exists $i_0 \in \{1, \dots, d\}$ such that, $x_i = y_i$ for all $i \neq i_0$, and $|x_{i_0} - y_{i_0}| = 1$). In other word starting from a point $x \in \mathbb{Z}^d$, we have the same probability to go from x to another neighbor of x.

Remark 7.3 A transition function a can also be defined as a function from V^2 to $[0, 1]$ by, for $(x, y) \in V^2$,

$$a_{x,y} = \begin{cases} a_{(x,y)} & \text{if } (x, y) \in E \\ 0 & \text{else.} \end{cases}$$

With this point of view, if V is finite, a defines a matrix A which corresponds to the adjacency matrix but with $a_{x,y}$ instead of ones.

Let Y_n be the law of X_n, i.e. the vector $(\mathbb{P}(X_n = x))_{x \in V}$. Then, if we compute Y_{n+1} the law of X_{n+1}, we observe that for a given $y \in V$, the probability for X to be in y at time $n + 1$ is the sum over all x neighbors of y of the probability for X to be in x at time

n and go to y. Thus, if we denote by A the matrix associated to a, we have the following

$$\begin{aligned}
\mathbb{P}(X_{n+1} = y) &= \sum_{x\sim y} a_{x,y}\mathbb{P}(X_n = y)\\
(Y_{n+1})_y &= (AX_n)_y\\
Y_{n+1} &= AX_n\\
Y_n &= A^n X_0
\end{aligned}$$

Definition 7.4. Let $G = (V, E)$ a graph, $f : V \to \mathbb{R}$ and a be a transition function on G. We define the *Laplacian* of f by $\Delta f : V \to \mathbb{R}$ as follows:

$$\Delta f : x \mapsto \sum_{y\sim x} a_{x,y}(f(y) - f(x))$$

In the following, we will consider only finite connected subgraph of $\mathbb{Z}^d$ (we can think to the box $\Lambda_n := \{-n, n\}^d$), that is, a finite set $V \subset \mathbb{Z}^d$ in which two vertices are adjacent if and only if $|x-y|_1 = 1$. We will denote Λ the set of vertices and $\Gamma = \partial\Lambda$ the set of vertices in Λ having a neighbor in $\mathbb{Z}^d$ which is not in Λ. Moreover, the transition probability we will consider here will be given, as in Example 7.2, by $a_{x,y} := \frac{1}{d(x)}$ (and in the expression of the Laplacian, $a_{x,y} := \frac{1}{2d}$).

7.2 Dirichlet problem

Definition 7.5. A function $f : \Lambda \to \mathbb{R}$ is called *harmonic* if, for all $x \in \Lambda \setminus \Gamma$,

$$\Delta f(x) = 0.$$

Proposition 7.6. *The following problem:*

$$\begin{cases} \Delta u = 0 \\ u_{|\Gamma} = f \end{cases} \tag{7.1}$$

admits a unique solution.

The uniqueness is a consequence of the *maximum principle*.

Lemma 7.7. Let $u : \Lambda \to \mathbb{R}$ an harmonic function. Then u reaches its maximum (its minimum too) on the set $\Gamma = \partial\Lambda$.

Proof. Suppose that u reaches its strict maximum at a point $x \notin \Gamma$. Then, in the sum $\Delta u(x) = \sum_{y\sim x}(u(y) - u(x))$, each term is strictly negative, which is absurd because of the harmonicity of u. □

Proof of Proposition 7.6. **Uniqueness:** let u and v be solutions of this problem. Then, $w := u - v$ is a solution of the following problem: $\begin{cases} \Delta w = 0 \\ w_{|\Gamma} = 0. \end{cases}$ Using Lemma 7.7, we observe that $w = 0$ i.e. $u = v$.

Existence: we define $\tau := \inf\{n \in \mathbb{N} \mid X_n \in \Gamma\}$ (almost surely, $\tau < +\infty$), and $u : x \mapsto \mathbb{E}_x[f(X_\tau)]$. We will show that u is solution to the problem.

It is clear that for $x \in \Gamma$, $\tau = 0$ under $\mathbb{P}_x$, thus $X_\tau = x$ and

$$u(x) = \mathbb{E}[f(x)] = f(x). \tag{7.2}$$

For a given x not in the boundary ($x \in \Lambda \setminus \Gamma$) we have

$$\mathbb{E}_x[f(X_\tau)] = \sum_{y \sim x} \mathbb{P}[x \to y]\, \mathbb{E}_y[f(X_\tau)] = \sum_{y \sim x} a_{x,y} \mathbb{E}_y[f(X_\tau)].$$

As, for every $x \in \Lambda$, $\sum_{y \sim x} a_{x,y} = 1$, we get that $\Delta(u) = 0$ and u is harmonic. □

7.3 Transience and Recurrence

In this section, we will consider infinite graphs $\mathbb{Z}^d$, where the transition probability is

$$a_{x,y} = \begin{cases} \frac{1}{2d} \text{ if } x \sim y \\ 0 \text{ else.} \end{cases}$$

The question asked is: if X_n is the simple random walk leaving at 0, do we have that $X_n = 0$ for an infinite values of n? Or equivalently, does the walk return to 0?

Definition 7.8. A walk is *recurrent* if almost surely, it visits each vertex infinitely often. The random walk is *recurrent* if and only if $\mathbb{E}_0[N(0)] = +\infty$ and *transient* in the other case.

Let fix a given $y \in \mathbb{Z}^d$. We define $N(y) := \sum_{n=0}^{+\infty} \delta_{X_n = y}$, i.e. the number of passages of the walk at the point y, where $\delta_{X_n=y} = 1$ if $X_n = y$ and 0 otherwise.
Then the function $G_y : x \mapsto \mathbb{E}_x[N(y)]$ is nearly harmonic. Actually, its Laplacian is equal to zero in every vertex except y, we have

$$\Delta G_y(x) = -\delta_y(x).$$

To study this function G_y, we introduce the Fourier transform.

Definition 7.9. For any $f : V \to \mathbb{R}$ with finite support (i.e. there exists a finite subset of V such that $f = 0$ outside this set), we define the *Fourier transform* of f as follow:

$$\widehat{f} : \xi \mapsto \sum_{x \in V} f(x) e^{-ix \cdot \xi}.$$

Example 7.10 Let δ_x be the function such that

$$\begin{cases} \delta_x(x) = 1 \\ \delta_x(y) = 0 \text{ if } y \neq x \end{cases}.$$

Then $\widehat{\delta_x}(\xi) = e^{-ix \cdot \xi}$. In the case $x = 0$, we have:

$$\widehat{\delta_0} = 1.$$

The following result gives a link between the Fourier transform of a function f and the one of Δf.

Proposition 7.11.

$$\widehat{\Delta(f)}(\xi) = \frac{1}{d}\sum_{i=1}^{d}(\cos(\xi_i) - 1)\widehat{f}(\xi)$$

And this other result gives an expression of f according to its Fourier transform.

Proposition 7.12 (Inversion Formula)**.** *Let $x \in V$. Then*

$$f(x) = \frac{1}{(2\pi)^d}\int_{[0,2\pi]^d}\widehat{f}(\xi)e^{ix\cdot\xi}d\xi$$

Proof.

$$\begin{aligned}\int_{[0,2\pi]^d}\widehat{f}(\xi)e^{ix\cdot\xi}d\xi &= \int_{[0,2\pi]^d}\sum_{z\in V} f(z)e^{-iz\cdot\xi}e^{ix\cdot\xi}d\xi \\ &= \sum_{z\in V} f(z)\int_{[0,2\pi]^d} e^{i(x-z)\cdot\xi}d\xi \\ &= \sum_{z\in V} f(z)\prod_{k=1}^{d}\left(\int_{[0,2\pi]} e^{i(x_k-z_k)\xi_k}d\xi\right) \\ &= (2\pi)^d f(x).\end{aligned}$$

□

We assume that $y = 0$ (by translation, we can generalize the result), then $\Delta G_0 = -\delta_0$, and using Proposition 7.11 with the remark (that $\widehat{\delta_0} = 1$)

$$\widehat{G}_0(\xi) = \frac{d}{\sum\limits_{k=1}^{d}(1-\cos(\xi_k))}.$$

Then, using Proposition 7.12,

$$G_0(x) = \frac{d}{(2\pi)^d}\int_{[0,2\pi]^d}\frac{1}{\sum\limits_{k=1}^{d}(\cos(\xi_k)-1)}e^{ix\cdot\xi}d\xi$$

Proposition 7.13. *X is transient $\Leftrightarrow d \geq 2$.*

Proof.

$$\begin{aligned} X \text{ is transient} &\Leftrightarrow \mathbb{E}_0\left[N(0)\right] < +\infty \\ &\Leftrightarrow \int_{[0,2\pi]^d}\frac{d\xi}{\sum_{k=1}^{d}(1-\cos(\xi_k))} < +\infty.\end{aligned}$$

Because $1 - \cos(\xi_k) \sim \xi_k^2$, near 0 we have $\sum_{k=1}^{d}(1-\cos(\xi_k)) \sim \frac{|\xi|^2}{2}$, so

$$X \text{ is transient} \Leftrightarrow \int_{\mathbb{R}^d}\frac{d\xi}{|\xi|^2} < +\infty$$

which occurs if and only if $d \geq 3$.

□

7.4 Exercises

Exercise 7.14. Let $d \in \mathbb{N}$ and $d \geq 3$. Let $f : \mathbb{Z}^d \to \mathbb{R}$ with finite support. Does it exist g such that $\Delta g = f$?

Exercise 7.15. Let $G = (V, E)$ a directed graph which is symmetric (i.e. if $(x, y) \in E$ then $(y, x) \in E$). For $f : V \to \mathbb{R}$, we define its gradient ∇f on the edges as follows:

$$\nabla f := \begin{cases} E \to \mathbb{R} \\ e = (x, y) \mapsto \nabla f(e) = f(y) - f(x) \end{cases}$$

Let $g : E \to \mathbb{R}$ be a given function. Under which (minimal) assumption does it exist $f : V \to \mathbb{R}$ such that $g = \nabla f$?

Exercise 7.16 (Gambler's ruin)**.** Two players A and B makes tosses with a non-biased coin. At each step, the winner earn 1 from the loser, and the game stops when one of them is ruined. We assume that the total amount in game is 4 euros. We called X_n the evolution of the capital of A and

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

1. Give a representation of X_n in term of random walk on a given graph.
2. Which vertices are transient? Recurrent?
3. We admit that $Q^n \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3/4 & 0 & 0 & 0 & 1/4 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 1/4 & 0 & 0 & 0 & 3/4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ What is the limit of the law of X_n?
4. What is the probability that A wins (depending on his initial amount)?
5. Let $G_y : x \mapsto \mathbb{E}_y [N(x)]$ i.e. the expectation of the number of times that A has x euros knowing that he started with y euros. Compute the values of $G_y(x)$ for $y = 1$ and $y = 2$ and conclude.

Appendix A

Equivalence relation

Definition A.1. Let E be a set. A *relation* on E is a subset of $R \subseteq E \times E$. For $x, y \in E$, x is in relation with y with respect to R, denoted xRy, if $(x, y) \in R$.

Example A.2 (a) "is equal to" or $=$ is a relation on any set.

(b) "is less or equal than" or $\leq$ is a relation on $\mathbb{Z}$ or $\mathbb{R}$.

(c) For $n \in \mathbb{N}$, define the relation $\equiv_n$ on $\mathbb{Z}$ as follow: $x \equiv_n y$ if and only if n divide $x - y$.

(d) In graph $G = (V, E)$, $\sim$ is a relation on V.

Definition A.3. Let E be a set and R be a relation on E.

(a) R is *reflexive* if xRx for all $x \in E$.

(b) R is *symmetric* if for all $x, y \in E$, xRy implies yRx.

(c) R is *transitive* if for all $x, y, z \in E$, xRy and yRz implies xRz.

A relation which is reflexive, symmetric and transitive is an *equivalence relation.*

Example A.4 (a) "is equal to" or $=$ is an equivalent relation and is the typical example of equivalence relation.

(b) "is less or equal than" or $\leq$ on $\mathbb{Z}$ or $\mathbb{R}$ is not an equivalence relation because it is not symmetric.

(c) $\equiv_n$ is an equivalence relation.

(d) $\sim$ is not an equivalence relation because it is not transitive nor reflexive.

Let E be a set and R be a relation on E. For $x \in E$, we set

$$R_x = \{y \in E \mid yRx\} \qquad \text{and} \qquad {}_xR = \{y \in E \mid xRy\}.$$

Proposition A.5. *Let E be a set, R be a relation and $x, y \in E$.*

(a) If R is symmetric, ${}_xR = R_x$.

(b) If R is an equivalence relation,

$$R_x \cap R_y = \begin{cases} R_x & \textit{if } xRy, \\ \emptyset & \textit{else.} \end{cases}$$

Proof. The proof is left to the reader as an exercise. $\square$

If R is an equivalence relation on a set E, $R_x = {}_xR$ is called the *equivalence class* of x. By Proposition A.5, we have a partition of E into equivalence classes.

Example A.6 (a) For a set E and $x \in E$, the equivalence class of x for the relation $=$ is $\{x\}$.

(b) For natural number x, the equivalence of x for the relation $\equiv_n$ is $\{x + kn \mid k \in \mathbb{Z}\}$.

Appendix B

Probabilities

Definition B.1. Let Ω be a set, called universe. A *probability* $\mathbb{P}$ on Ω is a mapping from $\mathcal{P}(\Omega)$ (the set of subsets of Ω) to $[0,1]$ satisfying the following conditions.

(i) $\mathbb{P}(\Omega) = 1$.

(ii) For $(A_i)_{i\in I}$ a collection of subset of Ω such that for all $i \neq j$, $A_i \cap A_j = \emptyset$, we have

$$\mathbb{P}(\bigcup_{i\in I} A_i) = \sum_{i\in I} \mathbb{P}(A_i).$$

Remark B.2 Actually, $\mathbb{P}$ could be define on a smaller subset of $\mathcal{P}(\Omega)$ under some assumption: a σ-field (class stable by countable union, complement and containing the empty set).

Definition B.3. A *random variable* is a map $X : \Omega \to \mathbb{R}$.

To simplify, we assume here that exists a finite (or countable) set I and a collection $(A_i)_{i\in I}$ of subset of Ω such that $X := \sum_{i\in I} a_i \delta_{A_i}$, where, for $A \subseteq \Omega$ and $y \in \Omega$, $\delta_A(y) = 1$ if $y \in A$ and 0 else. For exemple, this is true if X takes its value in a finite (or countable) subset of $\mathbb{R}$.

Definition B.4. Let X be a random variable. The *expectation* of X is the real number

$$\mathbb{E}[X] := \sum_{i\in I} \mathbb{P}(A_i) a_i.$$

The expectation could be think as the means of the values of X according to the probability $\mathbb{P}$.

Example B.5 Let X be a Bernoulli variable of parameter p, which we denote $X \sim \mathcal{B}(p)$ i.e. $X = 1$ with probability p and 0 with probability $1 - p$. What is the expectation of X?

Definition B.6. Let X and Y be two random variables taking values in a finite set $A = \{x_1, .., x_n\}$. X and Y are *independent* if for all $i, j \in \{1, 2, \ldots, n\}$,

$$\mathbb{P}[X = x_i \text{ and } Y = y_j] = \mathbb{P}[X = x_i]\,\mathbb{P}[Y = y_j]$$

where for $1 \leq j \leq n$, $(X = x_j)$ denotes the set $\{\omega \in \Omega \mid X(\omega) = x_j\}$.

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